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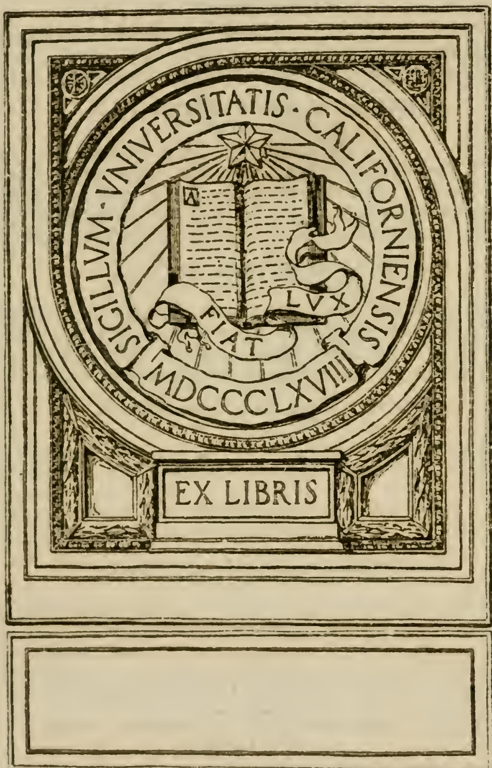
PRICE LIST, JANUARY 1, 1906.



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GEORGE W. JONES, Publisher,

No Agents. Ithaca, N. Y.

A DRILL-BOOK IN ALGEBRA

BY

PROF. GEORGE WILLIAM JONES

OF

CORNELL UNIVERSITY.

Sixth Edition.

ITHACA, N. Y.:
GEORGE W. JONES.
1906

PREFACE.

So far as concerns mathematical studies there are always two classes of pupils: those that will use mathematics later in life as one of the tools of their trade—the engineers, architects, accountants, teachers, scientists; those that will not so use it.

To both of these classes the careful study of the elements of geometry and algebra is important: to the first class as laying a sure foundation for work in the higher mathematics and for its professional applications; to the others as giving the power and the habit of exact statement and rigorous proof. In this later day we pride ourselves on our laboratories, in which the pupil comes face to face with the facts and forces of nature—every mathematical recitation-room under an able teacher is a laboratory in logic, and for sound logic there is always an unlimited demand.

This book is for use by the more advanced classes in the high schools and academies and the lower classes in the colleges; and its primary object is to teach young men and women to think. From the beginning the philosophy of the subject is made prominent; and in writing it the author set himself the double task of writing a book whose definitions should be precise and whose proofs should be rigorous, and of writing one so simple that any diligent pupil could read it easily.

But he has not confined himself to definitions and proofs: a large collection of questions and exercises has been added; and for a good understanding of the fundamental principles, and readiness in their use, quite as much reliance is placed on

the questions as on the text. It is hoped that by his effort to answer these questions the pupil will be early taught to think earnestly, to think independently, and to think hard.

Believing that the elements of plane geometry are at least as simple as those of algebra, the author has assumed some knowledge of that subject in pupils using this book, and he has not hesitated to use geometric illustrations where their greater concreteness seemed to give greater clearness.

This book was undertaken as an abridgment of Oliver, Wait, and Jones' Treatise on Algebra; and at first it was hoped that cutting out the more abstruse portions of the text and the harder examples would fit the larger work for the use of the preparatory schools; but after that excision many alterations were found necessary, and in the end the order of topics was changed, new lines of proof were adopted, new questions and examples were prepared, and the whole book was rewritten.

The author is indebted both to the writers on algebra, from whose works he has drawn freely, and to his associates at Cornell University, who have been unsparing in their kindly assistance. In particular he returns thanks to Professor Hathaway, who outlined the discussion of the combinatory properties of the simple arithmetic operations, that of measures and multiples, and that of incommensurable numbers; to Mr. John H. Tanner, who made the selections from the Treatise and prepared the first draft of the copy; to Miss Ida M. Metcalf, who spent half a year in giving form to the text and preparing the questions and exercises; and to Professors Oliver and McMahon, who have read the greater part of the book either in manuscript or in proof.

But with all the care he could exercise, he is conscious that many errors have crept in, and that there are many defects that only use in the class-room can bring to light. He will esteem it a great favor, therefore, if his fellow-teachers will tell him freely what they find wrong either in method or matter, in general plan or detail.

SUGGESTIONS TO TEACHERS.

THE author's aim has been to discuss subjects in the order of their logical dependence, so as to construct a continuous and irrefutable line of argument throughout; but it must be remembered that the logical order is not always the easiest or the most natural, and that not only are the claims of mathematical science to be satisfied, but the wants of the individual immature pupil are to be met. It may often happen, therefore, that deviation from the order of the book will be of advantage; and a subsequent review in the order laid down will then show the drift of the thought and the links in the chain of reasoning.

This point should be early and often impressed upon the pupil: that many simple theorems are stated and proved, not that he may be convinced of their truth, for that conviction can be reached by repeated experiment, but that a firm foundation may be laid for a logical structure of ever-increasing height and complexity. That is, they are not ends in themselves, but only useful tools for future work.

If parts of the text seem too abstruse for his pupils, or the questions too hard, the wise teacher will reserve such parts for a later reading, and he will choose for himself the order in which the topics shall be presented. For example, he may find it well to take up parts of the second chapter before finishing the first, or to set parallel lessons from the two chapters. Theory and practice may thus go hand in hand.

In this book general principles are stated formally, as in text-books on geometry, and illustrations and applications follow; but the living teacher may well reverse this order, and before setting a new topic in the book he may draw out the whole matter from the pupils' own mind, by careful questioning after the Socratic method, first in simple illustrations and then in general principles. Afterwards the pupil may read and explain the text, and answer the questions set down for exercises. The author recognizes the distinction between the office of the text-book and that of the teacher, and he

places the man above the book; but he has been taught by his own experience that a book is very useful. And what should the book contain? A treatise on any subject contains the whole body of knowledge on that subject, well digested and well arranged and indexed, so that the reader may find all that he seeks within its pages; but it need contain no exercises for pupils and no questions. A drill-book is more modest in its aims: it leaves out all that is not necessary to the main purpose; it presents the great principles in due order and in simple language, so that the pupil may read them again and again, and it sets him, under his teacher's guidance, to find out their applications; it suggests to him the best methods of work; it offers him lists of questions on which he may task himself, and grow strong in the exercise; it helps the teacher to cross-examine him; it serves as a standard to both teacher and pupil; and it saves them endless labor in the giving and taking of notes and in the preparation and copying of exercises.

It has been the author's good fortune to have a few bright young people come to him every summer for a more complete preparation in elementary mathematics, and he has thus kept fresh in mind the wants of beginners. He has found these pupils needing a regular and persistent drill both in the statement and proof of the fundamental principles and in their application; and he has written this book to meet their wants. He submits it respectfully to the judgment of his fellow-teachers, only asking that they neither adopt it nor reject it without a very careful examination; for while in most things it follows well-worn lines, in others it makes radical departures from the common usage.

An answer-book (not a key) is in preparation for the convenience of teachers; and the whole list of questions has been printed on cards for use in the class-room.

GEORGE W. JONES.

ITHACA, N. Y., May 3, 1892.

CONTENTS.

I. THE PRIMARY OPERATIONS OF ARITHMETIC

II. THE PRIMARY OPERATIONS OF ALGEBRA.

III. SIMPLE EQUATIONS.

IV. MEASURES AND MULTIPLES.

V. VARIATIONS, PROPORTION, INEQUALITIES, AND INCOMMENSURABLE NUMBERS.

VI. POWERS AND ROOTS.

VII. QUADRATIC EQUATIONS.

VIII. THE THREE PROGRESSIONS, INCOMMENSURABLE POWERS AND LOGARITHMS.

IX. PERMUTATIONS, COMBINATIONS AND PROBABILITIES.

PRIMARY NOTIONS FOR YOUNGER PUPILS.

WHILE there are many new things to be learned in algebra, there is nothing contradictory to what has been already learned in arithmetic, and there are many points of resemblance between these two sciences.

For example, the same signs of operation, of grouping, and of equality, are used, $+$, $-$, $\times$, $:$, $(\,)$, $=$; and fractions, powers, and roots have the same meaning, and are written in the same form.

The differences come largely from the frequent

USE OF LETTERS TO REPRESENT NUMBERS.

But there is nothing arbitrary or mysterious about this use; the letters, for the most part, are abbreviations for words, and sometimes they serve to make the statements more general than if numerals alone were used.

The reasoning is the same whether the numbers be expressed in words, in letters, or in figures.

For example, if n stand for a certain number, say a man's age, or the number of books in his library; then $2n$ stands for the double of this number, $3n$ for its triple, $\frac{1}{2}n$ for its half.

To make this clearer, the pupil may answer these

QUESTIONS.

1. What is the meaning of $4n$? of $6n$? of $\frac{2}{3}n$? of $3n-\frac{1}{2}n$? of $\frac{1}{2}(3n+9n-7n)$? of $3\sqrt{(7n+8n)}$? of $n:\frac{1}{3}n$?

If n stand for 60, what are the values of these expressions?

2. So, if x, y, z stand for three numbers, say the cost of an algebra, a reader, and a grammar, for what does $x+y$ stand? $x+y+z$? $x+y-z$? $5x+2y-3z$? $\frac{1}{2}(2x+y-z)$? $5x-4y-z$?

3. If $x=50$, $y=30$, $z=70$, find the values of the expressions written above, and of these:

$$x+\frac{y}{z},\quad \frac{x+y}{z},\quad \frac{x}{z}+y,\quad \frac{x}{z+y},\quad z+\frac{x}{y},\quad \frac{z+x}{y},\quad \frac{z}{y}+x.$$

4. Show the difference in meaning between the expressions:

$$12a-7b+3c \quad \text{and} \quad 12a-(7b+3c);$$

$$(a+b)\times(c+d) \quad \text{and} \quad a+(b\times \epsilon)+d;$$

and, if $a=4$, $b=3$, $c=2$, $d=6$, find their values.

The use of single letters to stand for words, and of signs to denote operations and relations, constitute

A KIND OF SHORT-HAND WRITING,

and the resulting brevity of expression is of great advantage in many ways. For example, compare these two statements:

1. I want the result of adding to a number three times itself, subtracting twice the number from the sum, increasing the remainder by five times, and by four times, the same number, and finally subtracting seven times the original number from the last sum.

2. $a+3a-2a+5a+4a-7a$; wherein a stands for the number, and the additions and subtractions are indicated by the signs $+$ and $-$.

In the last form the result, $4a$, *i.e.*, four times the original number, is seen at a glance.

So, let x, y stand for any two numbers, of which x is the larger, and express in algebraic form:

the sum of the numbers; their difference; their product;

the proper fraction got by dividing one by the other; the improper fraction;

the product of the smaller number by their difference;

the sum, the difference, and the product of their squares;

the squares of their sum, their difference, and their product;

the product and quotient of their sum by their difference;

the product of the squares of their sum and their difference;

the product of the sum and difference of their squares;

the sum and the difference of their cubes;

the cube of their sum and of their difference.

So, tell in words what these algebraic expressions mean:

$$2(x+y);\quad xy(x+y);\quad \tfrac{3}{4}(x^2-y^2);\quad x^3+y^3;\quad x^2y^2;$$

$$(x+y)^2;\quad \tfrac{1}{2}(x+y+6);\quad 10-xy;\quad \tfrac{5}{6}(2x+3y);\quad x^3y^3;$$

The examples above are cases of

TRANSLATION INTO ALGEBRAIC FORMS.

The pupil may also translate the answers to these questions:

1. At a dollars a pair what will 5 pairs of gloves cost? b pairs? $2b$ pairs? a pairs? $2a$ pairs?

How many pairs can be bought for c dollars? for l dollars?

2. An orchard contains r rows of t trees each, and each tree bears b barrels of apples: how many barrels of apples are raised? and what is their value at d dollars a barrel?

3. A man bought three books, paying a dollars for the first book, b times as much for the second as for the first, and for the third c times as much as for the other two: what was the cost of each book? of all of them?

4. A bill of groceries shows t pounds of tea at x cents a pound, s pounds of sugar at y cents, and c pounds of coffee at z cents: what is the whole cost in cents? in dollars?

5. At x dollars a yard, what will c yards of cloth cost? If the same cloth be sold for y dollars a yard, and something be gained, what relation has y to x? How much is gained on one yard? on the c yards?

6. A square field is f rods long: how wide is it? what is its area? how long a fence is needed to enclose it? how much more to divide it into four equal square fields? how much to divide it into four equal rectangular fields?

Note that the numerical values are not determined, only the relations between the length, breadth, area, length of fences, and so on, relations which hold good whatever number f stands for; and state these relations in words.

7. A can do a piece of work in a days, B in b days, C in c days: what part of the work can A do in one day? B? C? A and B working together? B and C? C and A? A, B, and C? What does $\frac{1}{a}$ stand for? $\frac{1}{b}$? $\frac{1}{c}$? $\frac{1}{a}+\frac{1}{b}$? $\frac{1}{b}+\frac{1}{c}$? $\frac{1}{c}+\frac{1}{a}$? $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$? $1:\left(\frac{1}{a}+\frac{1}{b}\right)$? $1:\left(\frac{1}{b}+\frac{1}{c}\right)$? $1:\left(\frac{1}{c}+\frac{1}{a}\right)$?

$1:\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$? $1-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$? $1-3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$?

8. If $a=16$, $b=12$, $c=6$, find the value of each of the expressions above; and find what part of the work remains undone after A, B, and C have worked together three days.

9. If the figures used in writing a two-figure number be a, b, the number is written $10a+b$: express half the number; the square root of it; a number that is c units less than this number; a number that is c units less than a fourth part of it; the number whose figures are b, a.

10. If a boy be y years old now, how old was he a year ago? two years ago? how old will he be f years hence? how old when his age is doubled? what is a third of his age? two thirds of what it will be h years hence?

11. A rectangular pile of wood is a feet long, b feet wide, and c feet high: how many square feet are there in the top? in one end? in one side? how many cubic feet in the pile? how many cords? what is its value at d dollars a cord?

12. If x be the larger of two numbers and d their difference, what is the smaller number? If x be the smaller number and d their difference, what is the larger?

13. If p be the product of two numbers and n be one of them, what is the other? If q be the quotient and n one number, what is the other?

14. If a bushels of wheat cost a dollars, what is the price of one bushel? If c bushels of this wheat sell for h dollars, what is gained or lost on one bushel? on the whole lot?

15. At p cents a yard what will it cost to plaster the walls and ceiling of a room a feet long, b feet wide, c feet high?

16. If a man can row w miles an hour in still water, what progress will he make rowing with a tide that runs t miles an hour? rowing against the tide?

17. A man has m miles to walk in h hours; he walks r miles an hour for the first b hours: how fast must he walk the rest of the way?

18. A merchant began business with a capital of a dollars; the first year he doubled his money; the second year he gained b times the original capital; the third year he lost l dollars and died, leaving c children; the cost of settling the estate was s dollars: how much did each child receive? Is it possible that there was nothing to divide?

19. A and B start from the same place and walk in the same direction, A at a miles an hour and B, b miles: how far apart are they at the end of an hour? at the end of h hours? Do you know from this statement which man walks the faster?

So, if they go in opposite directions?

So, if starting from two places k miles apart, they walk towards each other? if away from each other?

THE USE OF LETTERS IN SOLVING PROBLEMS.

Sometimes statements are made of such a nature that the actual value of an unknown number, at first represented by a letter, may afterwards be found.

For example, to find a number such that if its double and its quadruple be added to it, the sum shall be 63.

This problem is easily solved by arithmetic, as follows:

If to a number its double be added the sum is three times the number; and this sum increased by four times the number is seven times the number, which is 63. If then 63 be seven times the number sought, that number is a seventh part of 63, *i.e.*, the number is 9.

The words *the number* are used seven times in this statement, and the process may be shortened by writing some letter, as x, for these words and expressing the operations and relations by signs. In this symbolic language, $2x$ stands for the double of the number and $4x$ for its quadruple; and it is easy to translate into algebraic forms what the statement above gives in words:

a number	increased by	its double	and	its quadruple	gives	63;
x	$+$	$2x$	$+$	$4x$	$=$	63

i.e., $x+2x+4x=63$, $7x=63$, $x=9$, as before.

So, if the sum of the ages of father, mother, son, and daughter be 100 years, if the boy be twice as old as his sister, the mother four times as old as her son, and the father's age be three times the sum of his children's ages; these facts may be expressed in algebraic form by writing x for the girl's age, $2x$ for the boy's, four times $2x$, *i.e.*, $8x$, for the mother's, $3x$ for the sum of the children's ages, and three times $3x$, *i.e.*, $9x$, for the father's age; then $9x+8x+2x+x=100$, $20x=100$, $x=5$; and the girl is five years old, the boy ten, the mother forty, and the father forty-five.

So, if a man pay \$45 for a saddle and a bridle, and the saddle cost four times as much as the bridle; then x may stand for the cost of the bridle, $4x$ for that of the saddle, and the equation $4x+x=45$ expresses all the facts set forth in the statement of the problem. From this equation comes $5x=45$, $x=9$, $4x=36$, the values sought.

For further practice the pupil may choose some letter or other character (a star or the picture of a dragon-fly would serve just as well if as easily made and read) to stand for one of the unknown elements of the problem, express the other unknown elements as multiples or parts of the letter, translate the word-statements into algebraic forms, solve the resulting equations, and finally determine the values sought.

1. A and B together have \$100, and B has three times as much as A; how much has each of them?

2. A man paid \$24 for a hat, a vest, and a coat; the vest cost twice, and the coat three times, as much as the hat: what was the cost of each of them?

3. Divide 108 into three parts such that the second part shall be twice the first, and the third three times the second.

4. Three trees together bear 32 bushels of apples; the second tree bears twelve times as much as the first, and the third a fourth part of the yield of the second: how many bushels does the first tree bear?

5. John is three times as old as Henry, and the difference of their ages is 12 years: how old is each of them?

6. The difference of two numbers is seven times the less, and if four times the less be taken from the greater, the remainder is 24: what are the numbers?

7. B is three times as old, and C four times as old, as A, and the sum of their ages is s years: how old is each of them?

THE NATURE OF EQUATIONS.

In the problems that have been solved above, some number, or the difference of two numbers, or the sum of two or more numbers, has always been given as equal to some other number. Such expressions of equality are called *equations*, and their nature and uses should be well understood. The two parts of an equation, separated by the sign of equality, are its *members*, and the numbers that make up the two members and are joined by the signs $+$ and $-$ are its *terms*.

An equation of itself cannot be said to have any sign or value; it is like a balance, one member being the thing weighed and the other member the weights. The sign of equality corresponds to the pivot.

Regarding an equation in this light, it is evident that the same number may be added to, or subtracted from, both members of an equation, and that both members may be multiplied, or divided, by the same number, and the resulting numbers be still equal.

For example, if $2x=6$, then $x=3$ and $6x=18$.

So, if $x=6-2x$, then $x+2x=6$ and $x=2$.

So, if $x+3=24-2x$, then 3 may be subtracted from both members, and $2x$ be added, and there results the equation $3x=21$; then both members may be divided by 3 and $x=7$.

So, if $\frac{1}{4}x=3$, then $x=12$; and if $\sqrt{x}=3$, then $x=9$.

In the same manner the pupil may solve the equations below:

1. $4-2x=8-6x$. 2. $3x-48=x+12$. 3. $1=10-\frac{3}{4}x$.

4. $\frac{1}{2}(x-6)=20$. 5. $5+6x=2x+17$. 6. $\sqrt{x}-3=9$.

7. $3x=\sqrt{9}$. 8. $5(x-6)=\frac{1}{2}(17-x)$. 9. $\sqrt{x}+5=2\sqrt{x}+3$.

10. $\sqrt[3]{3x}-2=1$. 11. $\sqrt{x}-1=\sqrt[3]{27}$. 12. $\sqrt[3]{x}+1=\sqrt{16}$.

THE USE OF LETTERS AND EQUATIONS IN STATING RULES AND PRINCIPLES.

The familiar rules and principles of arithmetic may often be stated in algebraic language. One advantage of such statements is their brevity and clearness, as noted above, and the consequent ability of the pupil to see the whole at once, and to note the relations of all the parts.

For example, that *the value of a fraction is not changed if both its terms be multiplied by the same number*, a statement of two lines, is expressed in algebraic form by the use of six letters and three signs: it is written $\frac{a}{b}=\frac{a\times m}{b\times m}$, wherein $\frac{a}{b}$ stands for any fraction whatever and m for any multiplier.

So, the equation $\frac{a}{b}=\frac{a\div m}{b\div m}$ means that *both terms of a fraction may be divided by the same number without changing its value.*

So, the proposition—*if four numbers be in proportion, the product of the extremes is equal to the product of the means,* is expressed by writing if $a:b=c:d$, then $a\times d=b\times c$, wherein a, b, c, d, are any four numbers.

So, that *the remainder is not changed if both minuend and subtrahend be increased by the same number* is expressed by the equation $(m+n)-(s+n)=m-s=r$, wherein m is the minuend, s the subtrahend, r the remainder, n any number.

So, $\frac{a}{b}$ and $\frac{c}{d}$ may stand for any two fractions, and the rule for their addition may be briefly written $\frac{(a\times d)+(b\times c)}{b\times d}$.

The pupil may in like manner write the rules for subtracting one fraction from another; for multiplying two fractions together; for dividing one fraction by another.

So, he may express the relations between the dividend, divisor, quotient, and remainder, in division.

So, he may write down the well-known properties of proportion which are expressed by the words *alternation, inversion, composition and division.*

ALGEBRA.

I. THE PRIMARY OPERATIONS OF ARITHMETIC.

The pupil who begins the study of algebra is already familiar with the simpler operations of arithmetic, and has given some thought to the reasons for those operations. As algebra is but a larger arithmetic, he may well stop to review the fundamental principles, and try to fix in his mind more precise notions of what is to be the basis of his larger knowledge.

As distinguished from geometry, algebra treats of numbers and of numbers only; and this chapter, after the definition of number and the distinction between the different kinds of simple numbers, goes on to define and discuss the operations of multiplication and division, of addition and subtraction, and of involution and evolution.

Algebra differs from arithmetic chiefly in this: that while in arithmetic every number has a definite and fixed value, and the numerical expression of value is the thing always sought, in algebra the expression of the relation of numbers, or of some general principle, is often what is aimed at, the numerical computation being an arithmetic operation that may be performed later or omitted.

Algebra differs still further from arithmetic in the use of a set of symbols that constitute a language of its own, and among the advantages of this symbolic language are these: clearness, brevity, and generality of statement; the ability to mass directly under the eye, and thus to bring before the mind as a whole, all the steps in a long and intricate investigation; and the facility of tracing a number through all the changes it may undergo.

§ I. NUMBER.

The subject-matter of Algebra is number, and numbers come from counting and measuring ; they answer the questions *how many* and *how much*. Anything that is to be measured is a *quantity*, and the result of the measurement is a *number*.

E.g., if a boy count the apples in a basket the answer is a number, say twelve.

So, he may measure the side of a room with a yard stick and find how many times, say twelve, a single yard is contained in the whole length.

INTEGERS AND FRACTIONS.

If the things counted be entire units, or if the thing measured contain the measuring unit exactly so that there is no remainder, the number is an *integer;* but if the measurement can be completed only by using some part of the first unit as a new unit of measurement, the resulting number, for the portion so measured, is a *simple fraction*.

E.g., if a side of a room contain a yard twelve times with a remainder that contains a third part of a yard twice, the whole length is twelve yards and two thirds.

CONCRETE AND ABSTRACT NUMBERS.

In the examples above, the number is closely associated with the things counted or measured, and the whole answer to the question *how many* is twelve apples, or twelve and two-thirds yards. Such numbers are *concrete numbers*, and concrete numbers may be defined as measured quantities.

E.g., twelve apples, five days, twenty pounds.

But an *abstract number* implies some operation, and it may be called an *operator*. The simplest operations are those of repetition, such as doubling, tripling, and quadrupling; and of partition, such as halving and quartering.

E.g., if a child pick up twelve blocks, he may pick up one block, then another, and another till he has twelve of them,

and the operation that the operator twelve calls for is repetition of the single act of picking up a block.

So, he may part his blocks into two equal piles, or cut an apple into two equal parts; the operator is two and the operation is *partition.*

That thing which is acted upon is the *unit.*

E.g., the single block in the first case, and the whole group of blocks, or the apple in the other, is the unit.

The combination of a unit and a numerical operator forms a concrete number.

QUESTIONS.

1. Give some number that answers the question *how long a time,* the question *how much,* the question *how far.*

2. Name the different standard units of time, and state their relation to each other.

So. of length, of area, of volume, and of weight.

Of these units, which are natural? which artificial?

3. Can the question *how ill* be answered by giving a number?

So, of the questions *how difficult, how good, how beautiful?*

So, of the questions *how warm, how bright, how strong?*

4. By what unit can a room that is twelve and two-thirds yards long be measured, so as to give an integer result?

5. If the same thing be measured, what is the effect on the number if a smaller unit be used? if a larger unit?

6. Count two dozen eggs with six eggs as the unit; what is the operator? with one egg as the unit, what is the operator?

7. Express the distance six feet with three inches as the unit, then take a smaller unit and a smaller; how far can this process of reducing the unit be carried?

So, if the unit be taken larger and larger?

8. Draw a line six inches long, and express three quarters of it with half an inch as the unit. How many distinct units are here used? How many operations? how many operators?

Which of these operations are repetitions? which partitions?

EQUAL NUMBERS.

That two concrete numbers be *equal*, it is necessary that the things counted or measured be of the same kind, and that, if the units be taken of the same magnitude, there shall be as many units in the one group as in the other.

If there be more units in one group than in the other, the first number is *larger* than the other; if fewer, it is *smaller*.

That two abstract numbers be equal, it is sufficient that when operating on the same unit they shall give the same result; examples of such numbers appear later.

That two numbers are equal is shown by the sign $=$; that they are unequal by $\neq$; that the first is larger than the other by the sign $>$; that it is smaller by $<$.

EXPRESSION OF NUMBERS.

In algebra numbers are represented by Arabic numerals, as in arithmetic; they are also often represented by letters.

E.g., in questions about interest, p may stand for principal, r for rate, t for time, i for interest, a for amount.

If there be four promissory notes, the four principals may be writen p', p'', p''', p^{iv}, read *p prime, p second, p third, p fourth*, or p_1, p_2, p_3, p_4, read *p one, p two, p three, p four;* and the corresponding rates and times would then be written r', t', r'', t'', r''', t''', r^{iv}, t^{iv}, or r_1, t_1, r_2, t_2, r_3, t_3, r_4, t_4.

Letters or figures attached to other letters are *indices*.

It is customary to express a concrete number by writing the operator first, and after it the unit.

E.g., twelve blocks, half an apple, two thirds of a yard.

But the purposes of algebra are sometimes better served by writing the unit first and following it with the sign of operation and the operator.

E.g., block $\times 12$, wherein the cross means repetition.

So, apple : 2, wherein the colon means partition, or apple $\times 1/2$.

So, yard : 3×2, or yard $\times 2/3$, means that the yard is divided into three equal parts, and two of them are taken.

QUESTIONS.

1. Find other concrete numbers that are equal to three hundred minutes; to thirty-six inches; to half a mile.

2. If there be two farms each of one hundred acres, are they equal in area? have they the same shape? are they equal in value?

3. If there be two farms, one of fifty acres worth a hundred dollars an acre, and the other of a hundred acres worth fifty dollars an acre, in what respect are the two farms equal?

What two concrete numbers are now equal?

What is the unit, and what are the operators that give these two equal concrete numbers? are they found by precisely the same, or by different, operations?

4. One room is four yards by six and another three yards by eight; will the same carpet cover both floors? can it be made up in two parts so as to cover either floor at will?

In what respect are the two floors equal? in what unequal?

5. If the answer to the question *how long* be six days, which is thought of first, the unit day or the number six?

6. If the length of a room be sought, which is found first, the yard stick or the length in yards?

7. What does a salesman do when he measures off 50 yards of carpet? so, when he cuts this carpet into six equal breadths?

In each of these operations what is the unit and what is the operator?

8. Following the same system of notation as in interest, how can the area of a rectangular house-lot be expressed if the length and breadth be known in feet? How can the cost be expressed if the price per square foot be also given?

So, the cost of a block of marble whose length, breadth, and height are known, and the price per cubic foot?

9. Write an expression to show the order of procedure if a grocer sell ten eggs to each of two customers for three successive days; ten eggs to each of three customers for two days.

§ 2. MULTIPLICATION AND DIVISION.

The *product* of a concrete number, the *multiplicand*, by an abstract number, the *multiplier*, is the result of the repetition or partition of the concrete number by the other used as an operator. The product is a concrete number of the same kind as the multiplicand. Multiplication thus embraces halving and quartering as well as doubling, tripling, and quadrupling.

E.g., $\$5 \times 10 = \$50$, $\$50 : 10 = \$5$, or $\$50 \times 1/10 = \$5$.

The product of two concrete numbers is an absurdity.

E.g., the product $\$2 \times 5$ days is impossible;
but if a man earn \$2 a day, in 5 days he earns \$10,
i.e., $\$2 \times 5 = \$10$, wherein 5, not 5 days, is the multiplier.

The solution rests on the well-recognized relation between the time and the wages earned: as five days is five-fold one day, so the wages of five days is five-fold the wages of one day.

The *product* of two or more *abstract numbers* is an abstract number that gives the same result when operating on a unit as is obtained when the unit is multiplied by the first of the given numbers, the product so found multiplied by the second number, and so on till all the numbers are used. The numbers are *factors* of the product.

E.g., if of three men A, B, C, A has \$50, B twice as much as A, and C three times as much as B;
then B has $\$50 \times 2$, or \$100; and C has $\$100 \times 3$, or \$300,
i.e., $\$50 \times 2 \times 3 = \$300$;
but since $\$50 \times 6 = \$300$,
therefore the product of the two abstract numbers 2, 3, is the abstract number 6.

That the product of two or more factors is to be used instead of the factors in succession, may be expressed by enclosing them in brackets or placing them under a bar, and these signs indicate that the expression so enclosed is to be first simplified and then used as a single number.

E.g., $\$50 \times (2 \times 3)$ or $\$50 \times \overline{2 \times 3}$.

The factors so used form a *group* of factors.

QUESTIONS.

1. When is multiplication a process of repetition, and when of partition?

2. By what process other than multiplication can the product of five days by seven be found?

3. Can an abstract number be multiplied by a concrete number? Can an abstract number be made concrete by multiplication?

4. If a concrete number be multiplied by an abstract number, of what kind is the product? Can a concrete product be found without using a concrete multiplicand?

5. In finding the area of a rectangle we seem to multiply one concrete number, the length, by another concrete number, the breadth; state what is really done.

6. What is the test of equality for abstract numbers?

7. How are we convinced that the product of the abstract numbers 3 and 2 is the abstract number 6?

Does this reasoning apply to this particular pair of numbers only, or is it in the nature of a general proof that applies alike to every pair of abstract numbers? e.g., does it prove that the product of the two abstract numbers 3 and 5 is the abstract number 15?

8. What is the cost of 10 cases of eggs, each containing 6 boxes that hold two dozen, at $1\frac{1}{2}$ cents apiece? Exhibit the factors in the order named.

Exhibit them in order if the number of dozen be first found and the price per dozen.

So, if the cost of one case be first found.

9. What change is made in the product by multiplying the multiplicand by some number and then using the same multiplier as before?

So, by multiplying the multiplier and leaving the multiplicand unchanged?

10. Name the integer factors of 12, of 35, of 315.

MULTIPLICATION ASSOCIATIVE.

The sign ∵ means *since* or *because;* ∴, *therefore;* $\cdots$, *and so on;* and the letters Q.E.D., *which was to be proved.*

THEOR. 1. *The product of three or more abstract numbers is the same number, however the factors be grouped.*

E.g., let 5, 3, 1/2, 7, be four abstract numbers;

then are the products $(5 \times 3) \times (1/2 \times 7)$, $5 \times (3 \times 1/2 \times 7)$, $(5 \times 3 \times 1/2) \times 7$ each equal to $5 \times 3 \times 1/2 \times 7$.

For, to multiply a unit by the product 5×3 gives the same result as to multiply the unit first by 5 and that product by 3; [df. prod. abs. nos.

So, to multiply this product by the product $1/2 \times 7$ gives the same result as to multiply it first by 1/2 and that product by 7;

and the result of the four multiplications is the product unit $\times 5 \times 3 \times 1/2 \times 7$.

So, the product unit $\times (5 \times 3 \times 1/2 \times 7) =$ unit $\times 5 \times 3 \times 1/2 \times 7$.

and ∵ the abstract products $(5 \times 3) \times (1/2 \times 7)$, $5 \times 3 \times 1/2 \times 7$ do the same work when operating on a unit,

∴ these abstract products are equal;

and so for the other abstract products. Q.E.D.

So, to make the reasoning general, let a, b, $1/c$, $\cdots$ $1/k$, l be any abstract numbers, operators that mean repetitions and partitions;

then are the products $a \times (b \times 1/c) \times \cdots (1/k \times l)$, $(a \times b) \times (1/c \times \cdots 1/k \times l)$, and all others that may be formed by different grouping of the factors, each equal to the product $a \times b \times 1/c \times \cdots 1/k \times l$.

For, to multiply the concrete product unit $\times a$ by the abstract product $b \times 1/c$ is to multiply the concrete product unit $\times a$ by the abstract number b, and the consequent concrete product unit $\times a \times b$ by the abstract number $1/c$; [df. prod. abs. nos.

i.e., unit $\times a \times (b \times 1/c) =$ unit $\times a \times b \times 1/c$.

So, for the product of this product by the abstract products that follow in order,

i.e., unit $\times a \times (b \times 1/c) \cdots (1/k \times l)$,

$$= \text{unit} \times\ a \times b \times 1/c \cdots 1/k \times l$$

$$= \text{unit} \times (a \times b \times 1/c \cdots 1/k \times l).$$

So, for the other concrete products.

∴ the abstract product of these factors is the same, however they be grouped. Q.E.D.

The principle proved in theor. 1 is the *associative principle* of multiplication; the theorem is sometimes written, *multiplication is an associative operation.*

QUESTIONS.

1. By actual multiplication, show the equality of the products $\overline{5 \times 1/10} \times \overline{2 \times 7} \times \overline{1/14 \times 2}$, $\overline{5 \times 1/10 \times 2} \times \overline{7 \times 1/14 \times 2}$.

2. What is a theorem? what a proof? [consult a dictionary.

3. In the proof of theor. 1, for what purpose are the factors represented by letters? Which of the factors a, b, $1/c$, $\cdots$ $1/k$, l, indicate repetitions and which partitions?

4. Is the caption of theor. 1 a statement of whose truth we are certain at first, or one that must be proved?

5. The second paragraph of the proof "then are the products $\cdots$" is a restatement of the last line of the theorem; why this restatement? Are the statements that follow known to be true, or must they be proved?

6. May the proof of a theorem rest on statements that "seem reasonable," or must it rest on the authority of definitions and axioms, and of other theorems that have been already fully established by means of definitions and axioms?

7. Is the theorem true, and the proof conclusive, if there be but five factors? what is the meaning of the dots?

8. State the associative principle of multiplication.

9. Show that the statement of the theorem and its proof depend directly and wholly upon the definition of the product of abstract numbers.

10. MULTIPLICATION COMMUTATIVE.

THEOR. 2. *The product of two or more abstract numbers is the same number, in whatever order the factors be multiplied.*

(*a*) *Two factors, m, n, both expressing repetition;*

then the two abstract products $m \times n$, $n \times m$ are equal.

For let

$$\begin{array}{cccc}
* & * & * & * \ldots * \\
* & * & * & * \ldots * \\
\ldots & \ldots & \ldots & \ldots\ldots \\
* & * & * & * \ldots *
\end{array}$$

be a collection of like units, say stars, arranged in rectangular form, m units broad and n units deep, so as to form m vertical columns and n horizontal rows;

then ∵ the concrete product star $\times m$ is the m stars in one row,

∴ the concrete product star $\times m \times n$ is the stars in the n rows,

i.e., all the stars in the whole collection;

and ∵ the concrete product star $\times n$ is the n stars in one column,

∴ the concrete product star $\times n \times m$ is the stars in the m columns,

i.e., all the stars in the whole collection.

∴ the abstract product $m \times n$ does the same work as an operator on a unit as the abstract product $n \times m$;

∴ these two abstract products, $m \times n$, $n \times m$, are equal.

Q.E.D. [df. eq. abs. nos.

(*b*) *Two factors, m, 1/n, expressing a repetition, and a partition;*

then the two abstract products $m \times 1/n$, $1/n \times m$ are equal.

For, let the unit be divided into n equal parts, and let each part be represented by one of the n stars in any column taken from the block of stars shown above;

then ∵ in this block of stars the concrete product (one column $\times m$) is the stars in the m columns,

i.e., all the stars in the whole collection, consisting of n rows,

∴ the concrete product one column $\times m \times 1/n$ is the stars in one row;

and ∵ the concrete product one column $\times 1/n$ is one star,

∴ the concrete product one column $\times 1/n \times m$ is the stars in one row.

∴ the two abstract products $m \times 1/n$ and $1/n \times m$, when operating on the same unit, do the same work, and are equal. Q.E.D.

QUESTIONS.

1. In the block of stars used in theor. 2, how many of the m stars in a row are not shown? of the n stars in a column?

2. In the product star $\times m$ what is the unit? what is the operator? why is this product concrete?

3. What does the product star $\times m \times n$ represent? what the product star $\times n \times m$?

How do you know that these two products are equal?

Granting their equality, how does it follow that the two abstract products $m \times n$, $n \times m$ are equal?

4. In case (a) it is shown that the product of two abstract integers is the same, in whatever order the factors be taken; what relation does this truth bear to the complete theorem?

5. What is the effect of the operator m acting on a column of stars as a unit? of the operator $1/n$ acting on this product? What is the final result of the two operations?

6. What is the effect of the operator $1/n$ acting on a column of stars as a unit? of the operator m acting on this product? What is the final result of the two operations?

7. What single fact proves that the two abstract products $m \times 1/n$, $1/n \times m$ are equal?

8. Make a formal statement of the truth learned in case (b) as if it were a theorem by itself.

9. Show by a diagram that one seventh of six equal lines is equal to six sevenths of one of these lines.

(*c*) *Two factors,* $1/m$, $1/n$, *both expressing partition ;*

then the two abstract products $1/m \times 1/n$, $1/n \times 1/m$ are equal.

For, let the unit be divided into $m \times n$ equal parts, let each part be represented by a star, and let the whole be arranged in a block of m columns and n rows;

then ∵ in this block of stars, the concrete product block $\times\, 1/m$ is the stars in one column of n stars,

∴ the concrete product block $\times\, 1/m \times 1/n$ is one star;

and ∵ the concrete product block $\times\, 1/n$ is the stars in one row of m stars,

∴ the concrete product block $\times\, 1/n \times 1/m$ is one star,

∴ the two abstract products, $1/m \times 1/n$, $1/n \times 1/m$, when operating on the same unit, do the same work, and are equal. Q.E.D.

(*d*) *Three or more factors,* a, $1/b$, $c, \cdots 1/k$, l, *expressing repetitions and partitions ;*

then, however these factors are arranged at first in any product of them, that product is equal to their product when arranged in the order a, $1/b$, $c, \cdots 1/k$, l.

For in any one of these products, if a be not first it may be grouped with the factor before it, may change places with that factor without changing the whole product, and so come to be the first factor. [ths. 1, 2 (a, b, c).

So, $1/b$ may change places in turn with all the factors that stand before it except a, and so come to be the second factor, and so on ;

i.e., without changing the product, the factors may change places successively, and come to take the order a, $1/b$, $c, \cdots 1/k$, l.

∴ the abstract product $a \times 1/b \times c \times \cdots 1/k \times l$, and others that may be formed by different arrangements of the same factors, when acting upon any unit, do the same work, and are equal. Q.E.D.

The principle proved in theor. 2 is the *commutative principle* of multiplication, and the theorem is sometimes written, *multiplication is a commutative operation.*

QUESTIONS.

1. After the proof of cases (a, b) what must still be proved before theor. 2 is fully established?

2. In case (c) what kind of factors are considered?

3. What is the result when the operators $1/m$, $1/n$ act in this order on the group of stars? when the order is reversed?
What is proved by the two products' being the same?

4. As a separate theorem, state what is proved in case (c).

5. What relation does the product block $\cdot\, 1/m$ bear to the product block $\cdot\, 1/n$ when m is 4 and n is 3?

6. Show that a third of half a circle is half a third of it.

7. In what respect do the factors treated in case (c) differ from those in case (a)? in what are they like them?

8. How does case (d) differ from all the other cases?

9. In the proof of case (d) which of the statements made about the factor a rests on the authority of theor. 1?
Which of them on that of theor. 2?
With what kind of factors must a be grouped that case (b) may apply?

10. Show that a simple fraction with a numerator other than unity could, by the definition of a fraction, be regarded as the product of two factors of the form a and $1/b$, and that the proof in case (d) applies to such fractions.

11. State fully the truth proved in case (d).

12. What is the commutative principle of multiplication?

13. Show that the proof of theor. 2 depends mainly on the definition of equal abstract numbers.

14. By multiplication, show the equality of the products
$5 \times 1/10 \times 2 \times 7 \times 1/14 \times 4$, $\quad 2 \times 4 \times 1/10 \times 1/14 \times 5 \times 7$.
Arrange and group these factors in other ways, and show that all the products so found are equal.

COR. 1. *The product of two or more factors that express repetitions and partitions is an abstract simple fraction.*

For the factors that express partitions may be grouped together at the left,

and their product, showing how many equal parts the unit is divided into, gives the denominator of a fraction.

So, the factors that express repetitions may be grouped together at the right,

and their product, showing how many of these equal parts are taken, is the numerator of the fraction. Q.E.D.

COR. 2. *The product of two or more simple fractions is a simple fraction whose numerator is the product of the numerators of the factors; and whose denominator is the product of their denominators.*

COR. 3. *If n be any abstract integer, then the products $n \times 1/n$, $1/n \times n$ are each equal to unity.*

COR. 4. *If the numerator and denominator of a simple fraction be multiplied by the same integer, the value of the fraction is not changed thereby.*

For, let n/d be a simple fraction. and multiply n, d by a;

then $\because a \times 1/a = 1$,

$\therefore na/da = 1/d \times (1/a \times a) \times n = n/d.$ Q.E.D. [cr. 3.

RECIPROCALS.

Two abstract numbers whose product is unity are *reciprocals*.

E.g., $1/4$, 4; $3/2$, $2/3$; $3\frac{1}{2}$, $2/7$; 3×4, $1/12$; n/d, d/n.

THEOR. 3. *The product of the reciprocals of two or more abstract numbers is the reciprocal of their product.*

Let $a, b, \cdots 1/m, 1/n, \cdots p/q, r/s, \cdots$ be any abstract numbers;

then is the product

$$1/a \times 1/b \times \cdots m \times n \times \cdots q/p \times s/r \cdots$$

the reciprocal of the product

$$a \times b \times \cdots 1/m \times 1/n \times \cdots p/q \times r/s \cdots$$

For the product of these two products

$$= (a \times 1/a) \times (b \times 1/b) \times \cdots (1/m \times m) \times (1/n \times n) \times \cdots$$
$$(p/q \times q/p) \times (r/s \times s/r) \times \cdots$$
$$= 1 \times 1 \times \cdots = 1.$$

Q.E.D. [ths. 1, 2.

QUESTIONS.

1. What is a corollary?

2. Why must the product of factors expressing repetition or partition be abstract?

3. In what way does cor. 1 rest on theor. 2?

4. Is the product of two or more proper fractions larger or smaller than the several factors?

5. What is the office of the denominator of a fraction? What is the derivation of the word *denominator*, and what is its primary meaning? so, of the word numerator?

6. On what does cor. 2 rest, directly? indirectly?

7. By cor. 2 prove that $n \times 1/n = 1$, then by theor. 2 that $1/n \times n = 1$, and so that $n \times 1/n = 1/n \times n$.

8. In proving cor. 4, what use is made of the fact that $a \times 1/a = 1$?

9. What effect is produced by multiplying the numerator of a fraction 5/8 by six? the denominator by six? both numerator and denominator by six?

10. What is the time of a railway journey of 300 miles if the train run 10 miles an hour? if 20 miles? if 30 miles?

If the speed be multiplied by any number, by what number is the time multiplied?

11. If a certain piece of work is to be done, what is the relation between the time and the number of men employed?

12. What number is its own reciprocal?

13. What is the reciprocal of a proper fraction?

14. Without going through the whole proof of theor. 3, state why the product of the reciprocals of two or more numbers is the reciprocal of their product.

15. Can a concrete number have a reciprocal?

DIVISION.

Division is an operation that is the inverse of multiplication, i.e., when the product of two factors, and one of them, are given, division consists in finding the other factor. The product is now called the *dividend*, the given factor is the *divisor*, and the factor sought is the *quotient*.

If the dividend be concrete there are two cases of division:

1. The divisor abstract and the quotient concrete.
2. The divisor concrete and the quotient abstract.

The first is a case of partition, more or less complex, and the other a case of finding how many times one number is contained in another number of the same kind, a *ratio*.

E.g., ∵ the product of \$5 by 4 is \$20,

∴ the quotient of \$20 by 4 is \$5,

and the quotient of \$20 by \$5 is 4.

The sign of division is : or /, and the order of writing is dividend : divisor = quotient, or dividend/divisor = quotient. E.g., \$20 : 4 = \$5, \$20 : \$5 = 4, \$20/4 = \$5, \$20/\$5 = 4.

Theor. 4. *If the product of two factors be multiplied by the reciprocal of one of them, the result is the other factor.*

For, let f, g stand for the two factors, and p for their product;

then $\because p=f\times g$, [df. prod.

$\therefore p\times 1/f=f\times g\times 1/f=g\times f\times 1/f=g$. Q.E.D.

If the divisor be abstract, theor. 4 may be written, *the product of the dividend by the reciprocal of the divisor is the quotient.*

Cor. *If there be a series of multiplications and divisions, the final result is the same, in whatever order they be performed, and however the elements be grouped.*

For, every division by an abstract number is a multiplication by the reciprocal of the divisor,

and these multiplications may be performed in any order, and the factors be grouped in any way.

In applying this corollary, the pupil must take care lest he change the office of any element and multiply by it where he should divide, or divide by it where he should multiply.

E.g., $12 : 3 \times 2 \times 4 : 8 = (12 : 3 \times 4) \times (2 : 8)$
$\neq (4 : 3 \times 12) : (8 \times 2)$.

QUESTIONS.

1. Of the three elements of a division, can all be concrete? two concrete and one abstract? two abstract and one concrete? all abstract?

If a man earn $4 a day, in what time will he earn $20?

Are not the dividend, divisor, and quotient all concrete?

2. In each of the two cases of division shown on the opposite page, to what does the divisor correspond in the multiplication of which the division is the inverse?

3. If the dividend be multiplied by some integer and the divisor be unchanged, what is the effect on the quotient? if the divisor alone be multiplied? if both be multiplied by the same integer? if the dividend alone be divided? if the divisor alone be divided? if both be divided by the same integer?

State these principles as applied to the terms of a fraction.

4. Is a ratio a concrete or an abstract number?

5. On what authority is it said, in the proof of theor. 4, that $f \times g \times 1/f = g \times f \times 1/f$? on what that $g \times f \times 1/f = g$?

6. Can the second form of stating theor. 4 be used when the divisor is concrete? does the first form always apply?

How is the second form applied in the division of fractions?

7. Why may the multiplications and divisions spoken of in the corollary of theor. 4, be performed in any order?

Why may the factors be grouped in any way?

8. Replace the word *multiplied* by *divided* in cor. 4 theor. 2 and prove the resulting statement. What useful applications has this corollary in reducing fractions?

9. Prove that the quotient of the reciprocals of two numbers is the reciprocal of their quotient.

§ 3. POSITIVE AND NEGATIVE NUMBERS.

Sometimes things that are measured by the same unit are of opposite qualities.

E.g., assets and liabilities are both measured by the unit dollar.

So, dates A.D. and dates B.C. are both given in years.

So, the readings of a thermometer above and below zero are given in degrees.

In all such cases if the measuring unit be taken in the same sense as the thing measured, the resulting concrete number is *positive;* if taken in the opposite sense the number is *negative.* In which sense the unit shall be taken is a matter of custom, or of convenience.

E.g., the unit *dollar* may be taken either as a dollar of assets or as a dollar of liabilities:

if as a dollar of assets, then assets are positive numbers and liabilities are negative numbers;

if as a dollar of liabilities, then liabilities are positive and assets are negative.

So, if distances measured towards the north be positive, distances to the south are negative,

i.e., if the measuring unit be a northerly unit, southerly distances are expressed by negative numbers.

Some things admit of negatives and some do not.

E.g., time may be counted backwards as well as forwards from a given date.

So may distances from a given point.

So may heat and cold from an arbitrary zero.

So may money of account, as above.

But with real dollars, say five of them, the pupil will find when he tries to count past none—five, four, three, two, one, none—that he is trying to do what is impossible.

So, a negative number of persons by itself is an absurdity.

The primary notion of a negative concrete number is that of one which, when taken with a positive number of the same kind, goes to diminish it, cancel it altogether, or reverse it.

E.g., liabilities neutralize so much of assets, thereby diminishing the net assets, or leaving a net liability.

QUESTIONS.

1. Show that the measuring unit of longitude may be taken in either of two senses, and that whichever way the unit be taken the two kinds of longitude may be called positive longitude and negative longitude.

How are these two kinds of longitude now distinguished?

2. If positive longitude be taken in the direction of the sun's apparent motion and Greenwich be the starting point, is St. Petersburg in positive or negative longitude?

So, if the direction of the earth's rotation be taken as that of positive longitude?

3. If \$1 be taken as the unit of money possessed, how must money spent be represented? money inherited? money given away? money owed? money earned? money staked on a wager?

4. If distances toward the north be taken positive, how must the latitude of Morocco be expressed? of the equator?

5. What is the greatest possible positive latitude? the largest negative latitude?

6. What effect has a negative concrete number when combined with a larger positive number of the same kind? if the negative number be just as large as the positive? if larger?

7. If a wreck be acted on by a current setting northward and by a north wind, are the two forces of the same nature?

What determines the direction in which the wreck moves?

8. In rowing up a river that flows 4 miles an hour, what progress is made by a man who can row 4 miles an hour in still water? What effect then does the man's rowing produce?

Show that there are two changes, either of which would make his progress visible.

EXPRESSION OF POSITIVE AND NEGATIVE NUMBERS.

When denoted by Arabic numerals, positive numbers are written with the sign + or with no sign, and negative numbers with the sign −, before them. But if a number be denoted by a letter, it is not evident upon its face whether that letter denotes a positive or a negative number.

E.g., if the measuring unit be a dollar of assets, then +100, or simply 100 without the sign, means $100 of assets, and −100 means $100 of liabilities.

But N might stand either for +100 or for −100 at pleasure.

E.g., if N stand for +100, then −N stands for −100;

and if N stand for −100, then −N stands for +100.

In this use of the signs + and − they are *signs of quality.*

These signs are also used to indicate the operations of addition and subtraction, and they are then *signs of operation.*

To avoid confusion in the two uses of the same signs, the signs of quality may be made small and placed high up.

E.g., $^{+}100$ means $100 of assets, and $^{-}100$, $100 of liabilities.

But these small signs are used with the express understanding that $^{+}$ is attached only to numbers that are essentially positive, and $^{-}$ to those that are essentially negative. In that respect they may have a meaning that differs from the meaning of the large signs +, −.

There is a third sign, ±, made up of the two, and read *plus or minus;* if written ∓, it is read *minus or plus.*

E.g., ±3 is only an abbreviated way of writing the separate expressions +3 and −3; and $^{\pm}7$ is $^{+}7$ or $^{-}7$.

AN ABSTRACT NEGATIVE NUMBER AS AN OPERATOR.

As an operator an abstract negative number has two offices:

1. The repetition or partition of the multiplicand.
2. The reversal of its quality;

and every such number may be regarded as itself the product of two factors:

1. The absolute magnitude of the number, the *tensor*.
2. $^-1$, the *versor*.

E.g., $^-10 = {^+10} \times {^-1} = {^-1} \times {^+10}$.

So, \$1 assets $\times\, ^-10 =$ \$10 debts, and \$1 debts $\times\, ^-10 =$ \$10 assets.

So, 20 north-miles $\times\, ^-10 = 200$ south-miles,
and 20 south-miles $\times\, ^-10 = 200$ north-miles.

QUESTIONS.

1. If $a = -3$, what is the value of $3a$? of $-12a$?

2. Is it possible, in the course of an example, to have the expression -12 men? $^-12$ men?

Explain the difference between these two expressions.

3. What use is made of the signs $+$ and $-$ in both arithmetic and algebra? what use of them is peculiar to algebra?

4. If distances eastward from the point where we stand and time after the present moment be positive, and if a passing train be running eastward at 20 miles an hour: show that

in 5 hours it will be 100 miles east of us, $\quad {^+20} \times {^+5} = {^+100}$;
5 hours ago it was 100 miles west of us, $\quad {^+20} \times {^-5} = {^-100}$.

But if the train be running westward, show that

in 5 hours it will be 100 miles west of us, $\quad {^-20} \times {^+5} = {^-100}$;
5 hours ago it was 100 miles east of us, $\quad {^-20} \times {^-5} = {^+100}$.

5. Show that multiplication by a positive integer is a case of addition. What relation exists among the numbers added?

Then, with a positive multiplier, what relation has the sign of the product to that of the multiplicand?

6. If multiplication by a negative integer be regarded as a case of subtraction, what are the successive subtrahends?

How do their signs change in the process? what relation has the sign of the product to that of the multiplicand?

7. What two pairs of signs in the factors make the product positive? what two make it negative?

8. Show that the product of any even number of negative factors is positive, and that a product can be negative only when it contains an odd number of negative factors.

§ 4. ADDITION AND SUBTRACTION.

If the concrete numbers added be integers and simple fractions, then, at bottom, *addition* is but counting, either by entire units or by equal parts of a unit: on (forward) if positive numbers be added, off (backward) if negative numbers be added. The result is the *sum*, and the sign is +, read *plus*.

E.g., 50 cts. + 60 cts. + 90 cts. = \$2.

The *sum of* two or more *abstract numbers*, operators, is an operator that, acting on a unit by repetition, partition, or reversion, gives the same concrete number for result as if the several operators acted in turn upon the unit, and their results, like concrete numbers, were then added.

E.g., $7+5=12$;

for if a man earn \$2 a day, then the sum of his earnings for 5 days and for 7 days is his earnings for 12 days;

i.e., $\$2\times 5+\$2\times 7=\$2\times 12$.

So, $1/5=2/15+1/15$:

for if a field be divided into fifteen equal house-lots, a purchaser that takes two lots and one lot has a fifth part of the whole field; and that whether the three lots lie together or apart.

In algebra the word *addition* is used in a broader sense than in arithmetic, and covers negative as well as positive numbers.

E.g., he that has \$10,000 assets and \$4000 debts is worth but \$6000,

i.e., \$10,000 assets + \$4000 debts = \$6000 net assets,

and $^{+}10{,}000+{}^{-}4000={}^{+}6000$.

Though the numbers to be added must always be of the same kind, they are often expressed by letters whose values are not known, or in units whose values are different, or which cannot even be reduced to one sum: such a group is a *polynomial*, and the numbers to be added are its *terms*.

E.g., $5^h\,33^m\,35^s+12^h\,47^m\,25^s=18^h\,21^m$; in interest $a=p+i$.

QUESTIONS.

1. By dividing up a line, find the sum of $\frac{2}{3}$ and $\frac{1}{4}$ of it.
Hence find the sum of the two abstract numbers $\frac{2}{3}$ and $\frac{1}{4}$.
In what common unit are the two fractions expressed?

2. Before two numbers can be added how must they be expressed?

3. What is the nature of an operator that acts on a unit by repetition? by partition? by reversion? Name some operator that acts in two of these ways; in all three of them.

4. If a unit be acted on by two or more operators, why must the results of the several operations be concrete numbers of the same kind?

5. What two arithmetic operations may be indicated by the word *addition* in algebra? If negative numbers be added are the minus signs signs of quality or of operation?

6. If 240 men vote for a candidate and 160 vote against him, what is the sum of the votes he receives, or his majority?
Solve, first using signs of quality; then, stating the question differently, solve, using signs of operation only.

7. When can the addition of numbers be indicated but not performed?
What is the sum of $a, b, c,$ when their values are not known? when their values are 2, 3, 5?

8. In the expression $4\frac{3}{4}$, what sign of operation is understood between the integer 4 and the fraction $\frac{3}{4}$?
So, between the dollars and cents of \$18.50.
So, in the compound number $12^d\ 15^h\ 35^m\ 20^s$?
Show how the operations so indicated may be performed.

9. Can the sum of two numbers be smaller than one of them? smaller than each of them?

10. Draw a straight line and mark A, B, two points taken at random upon it; then show that the sum $AB + BA$ is naught, whichever direction be taken as positive.
So, take three points A, B, C, in any order upon a straight line, and show that $AB + BC = AC$ and that $AB + BC + CA = 0$.

ADDITION COMMUTATIVE AND ASSOCIATIVE.

Theor. 5. *The sum of two or more abstract numbers is the same number, in whatever order the numbers be added and however they be grouped.*

For, let ^+a, ^-b, $c, \cdots h/k$, $^{\pm}l$ be any abstract numbers, positive or negative,

let these numbers act as operators upon any unit,

and let the results be grouped and added in any way;

then ∵ the whole collection of units and parts of units is the same, whichever unit, group of units, part, or group of parts, be counted first, whichever second, and so on,

∴ the several sums of these operators do the same work and are all equal. [df. sum abs. nos., p. 22.

The principles here established are the *commutative* and the *associative principles of addition;* and theor. 5 may be written, *addition is a commutative and associative operation.*

If the pupil will cut card-board into any of the common geometric figures, say triangles and squares, and join them all together by placing them edge to edge in various ways, he will find that the figures resulting from this geometric addition are quite different in form, and that, in the sense of geometric equality, they are not equal at all; but he will find the areas, the numerical results of measurement, to be all equal: i.e., in the geometric sense the figures are unequal; in the algebraic sense, and for the purposes of algebra, they are equal.

Theor. 6. *The sum of two or more simple fractions is a simple fraction, or an integer.*

For, let a/b, c/d, $h/k \cdots$ be simple fractions;

then ∵ $a/b = ad/bd = 1/bd \times ad$, $c/d = bc/bd = 1/bd \times bc$, [th. 2, cr. 4, df. sim. frac.

∴ $a/b + c/d = 1/bd \times ad + 1/bd \times bc = 1/bd \times (ad + bc)$
$= (ad + bc)/bd$. [df. sim. frac., df. ad.

the sum of this sum and h/k is a simple fraction; and so on.

E.g., $2/3 + 3/4 = 17/12$; $1/4 + 6/8 = 1$.

QUESTIONS.

1. What is the associative principle of addition ?

What is the commutative principle?

2. Do the letters $a, b, c, \cdots$ stand for the same numbers in theor. 5 as in theor. 1? Can you tell which of these numbers are positive and which negative?

3. In the proof of theor. 5, is the value of the entire collection of units found by adding the results given by the several operators, or by the more detailed process of counting?

Which of the operators are to be applied by counting off?

Which, by partition and a later counting?

4. What other operation is commutative and associative?

5. If s stand for a square, t for a triangle, c for a half-circle, and r for a rhombus, is $\overline{s+r}+\overline{c+t}$ the same as $\overline{c+s}+\overline{r+t}$?

If the letters stand for the areas of the figures are the two sums equal?

6. Cut a triangle from paper with sides of different lengths, join the mid-points of the sides, and, cutting along these lines, divide the triangle into four triangles that may be shown to be all equal by placing one upon another; combine these triangles in all possible ways: are the geometric figures so found equal? are their areas equal?

7. If a merchant have various bills to pay from a sum of money lying before him, show that he has the same amount left after the bills are all paid, whatever be the order of their payment. Of what principle is this an example ?

8. If when two fractions are added the denominator of the sum be a factor of the numerator, how can the fraction be more simply written?

Is there any number that cannot be written in fraction-form ?

9. In the proof of theor. 6, if the sum of the operators ad, bc act on a unit, what is the result?

What is the result if ad, bc act separately on the unit and the results be then added?

What relations have these two results to each other?

MULTIPLICATION DISTRIBUTIVE AS TO ADDITION.

THEOR. 7. *The product of the sum of two or more abstract numbers by another number is the sum of the products of the first numbers by the other.*

Let $a,\ {}^-b,\ c, \cdots h/k,\ l,\ m$ be any abstract numbers;

then $(a+{}^-b+c+\cdots+h/k+l)\times m$

$=a\times m+{}^-b\times m+c\times m+\cdots h/k\times m+l\times m.$

For the product $(a+{}^-b+c+\cdots+h/k+l)\times m$

$=m\times(a+{}^-b+c+\cdots h/k+l)$ [th. 2, th. 6.

$=m\times a+m\times{}^-b+m\times c+\cdots m\times h/k+m\times l$

$=a\times m+{}^-b\times m+c\times m+\cdots h/k\times m+l\times m.$ [th. 2.

COR. *The product of two or more polynomials is the sum of the several products of each term of the first factor by each term of the second factor by each term of the third factor, and so on.*

For the product of two factors is the sum of the partial products of each term of one factor by each term of the other,

and the product of this product by a third factor is the sum of the partial products of each term of this product by each term of the third factor, and so on. Q.E.D.

The principle here established is the *distributive principle* of multiplication ; and theor. 7 is sometimes written, *multiplication is distributive as to addition.*

But addition is not distributive as to multiplication.

E.g., $(3+2)\times 5=(3\times 5)+(2\times 5)$; $(3\times 2)+5\neq(3+5)\times(2+5)$.

OPPOSITES.

Two numbers whose sum is 0 are *opposites* of each other.

E.g., ${}^+4,\ {}^-4$; ${}^-2/3,\ {}^+2/3$; ${}^-3\times{}^-4,{}^-12$; $3+4,\ {}^-7$; $3-4,\ 4-3$.

THEOR. 8. *The sum of the opposites of two or more abstract numbers is the opposite of their sum.*

Let ${}^+a,\ {}^+b,\cdots,\ {}^-m,\ {}^-n,\cdots,\ +p/q,\ -r/s,\cdots,$ be any abstract numbers,

then is the sum ${}^-a+{}^-b+\cdots+{}^+m+{}^+n+\cdots-p/q+r/s\cdots$ the opposite of the sum

${}^+a+{}^+b+\cdots+{}^-m+{}^-n+\cdots+p/q-r/s\cdots$

For the sum of these two sums

$$=({}^+a+{}^-a)+({}^+b+{}^-b)+\cdots+({}^-m+{}^+m)+({}^-n+{}^+n)$$
$$+\cdots+(p/q-p/q)+(-r/s+r/s)$$
$$=0+0+\cdots=0. \qquad \text{Q.E.D.} \qquad [\text{df. opp., th. 5.}$$

QUESTIONS.

1. In the proof of theor. 7 how many abstract numbers are used? are these numbers integers or fractions? are they positive or negative? is m abstract or concrete?

2. By what principle are the two products equal: $(a+{}^-b+c+\cdots+h/k+l)\times m$, $m\times(a+{}^-b+c+\cdots+h/k+l)$?

By what principle is the second product expanded into $m\times a+m\times{}^-b+m\times c+\cdots+m\times h/k+m\times l$?

By what, is this last product changed to the form sought?

3. Find the product of the two polynomials $a-b+c+\cdots-h/k+l$ and $m+1/n-p$, and show that their product is the sum of the partial products got by multiplying each term of the multiplicand in turn by each term of the multiplier.

Does the order in which the terms are multiplied affect the final product?

4. Find the product of $a+b$, $c+d$, $e+f$, and show that each term of this product contains three letters: one from the first factor, one from the second, and one from the third.

So, for the three factors $a-b$, $c-d$, $e-f$.

5. In the case of two opposite numbers, what is true of the two measuring units? of the number of times the unit is repeated in each of them? of the quality of the numbers as to that of the unit? of the tensors in the abstract operators? of the versors?

6. What number is its own opposite?

7. How is the sum of the opposites of two or more numbers related to the sum of the numbers? the product of the opposites of two numbers? that of the opposites of three numbers?

8. Explain the dependence of theor. 8 on theor. 5.

SUBTRACTION.

SUBTRACTION is an operation that is the inverse of addition, i.e., when the sum of two numbers, and one of them, are given, subtraction consists in finding the other number. The sum is now called the *minuend,* the given number is the *subtrahend,* and the number sought is the *remainder.*

The sign of subtraction is $-$, read *minus* and the order of writing is minuend $-$ subtrahend $=$ remainder; e.g.,

$$\begin{array}{lll}
\$50 - \$40 = \$10, & \text{for } \$40 + \$10 = \$50, & {}^+50 - {}^+40 = {}^+10; \\
\$40 - \$50 = {}^-\$10, & \text{for } \$50 + {}^-\$10 = \$40, & {}^+40 - {}^+50 = {}^-10; \\
{}^-\$50 - {}^-\$40 = {}^-\$10, & \text{for } {}^-\$40 + {}^-\$10 = {}^-\$50, & {}^-50 - {}^-40 = {}^-10; \\
{}^-\$40 - {}^-\$50 = \$10, & \text{for } {}^-\$50 + \$10 = {}^-\$40, & {}^-40 - {}^-50 = {}^+10; \\
\$40 - {}^-\$50 = \$90, & \text{for } {}^-\$50 + \$90 = \$40, & {}^+40 - {}^-50 = {}^+90; \\
{}^-\$50 - \$40 = {}^-\$90, & \text{for } \$40 + {}^-\$90 = {}^-\$50, & {}^-50 - {}^+40 = {}^-90.
\end{array}$$

THEOR. 9. *If to the sum of two numbers the opposite of one of them be added, the result is the other number.*

For let r, s stand for the two numbers and m for their sum,
then $\because m = r + s$, [hyp.

$$\therefore m + (-s) = r + s + (-s) = r + (s - s) = r. \qquad \text{Q.E.D.}$$

Theor. 9 may be written, *the sum of the minuend and the opposite of the subtrahend is the remainder.*

Hence the terms of an expression enclosed in a parenthesis and preceded by a minus sign can be added to like terms, provided the sign of every term so enclosed be first reversed.

COR. *If there be a series of additions and subtractions, the final result is the same, in whatever order they be performed, and however the elements be grouped.*

For every subtraction is an addition of the opposite of the subtrahend, and these additions may be performed in any order and the terms be grouped in any way.

In applying this corollary the pupil must take care lest he change the office of any element and add it where he should subtract or subtract it where he should add.

E.g., $12 - 3 + 2 + 4 - 8 = (12 - 3 + 4) + (2 - 8)$

$$\neq (4 - 3 + 12) - (8 + 2).$$

QUESTIONS.

1. May the three numbers involved in the process of subtraction be all abstract? all concrete? part abstract and part concrete?

If all be concrete what else must be true of them?

2. Interpret the examples given under the definition of subtraction, when the positive numbers stand for assets or earnings and the negative numbers for debts and expenses, and show what is meant by subtracting a negative.

3. In arithmetic, how does the remainder compare in size with the minuend? is this always true in algebra?

4. If from $a-b$, $c+d$ was to be subtracted, and c alone has been subtracted, is d to be added to this remainder or subtracted from it?

Prove that $(a-b)-(c-d+e)=a-b-c+d-e$.

5. If a polynomial be enclosed in a parenthesis and preceded by a minus sign, what changes must be made in removing the parenthesis?

So, if a parenthesis is to be inserted after a minus sign, what changes must be made in the signs of the terms included in it?

6. Simplify $3a-(a-4b+2a)$.

7. So, $x-2y-[{}^{-}2x+({}^{-}y-2x)-4x]$.

8. The rule for algebraic subtraction is: to each term of the minuend add the opposite of the like term of the subtrahend; what is the origin of this rule?

9. From the remainder subtract the opposite of the subtrahend; what is found?

What other operation will give the same result?

10. What two processes have been proved commutative and associative?

What other two processes may always be so indicated as to be examples of these two? with what caution?

11. Show in what respects theors. 4 and 9 are alike and in what they differ.

§ 5. INVOLUTION AND EVOLUTION.

The continued product of a number by itself is a *power* of that number. The number whose power is sought is the *base*, and the operator that shows how many times the base is used as a factor is the *exponent*; it is written at the right and above the base. *Involution* is the process of finding powers.

E.g., $4\times4\times4=4^3=64$, $1/4\times1/4\times1/4=1/64$. [th. 2, cr. 2.

In both examples the base is 4; in the first the operation is a continued repetition by 4, and in the other it is a continued partition by 4, operations that tend to neutralize each other; and this relation may be expressed by writing them 4^{+3} and 4^{-3}, wherein the positive exponent shows how many times 4 is used in repetition and the negative exponent shows how many times 4 is used in partition.

As an exponent, $^-1$ reverses the quality of the base, i.e., if the base denote repetition, the exponent $^-1$ changes the operation to partition, so that $n^{-1}=1/n$.

The words *positive* and *negative* as applied to powers refer to the exponents only; and *integer powers* are powers whose exponents are integers.

E.g., $(^-4)^3$ is a positive integer power, although its value, $^-64$, is negative.

So, 4^{-3} is a negative integer power, although its value, 1/64, is positive and a fraction.

PRODUCT OF INTEGER POWERS OF THE SAME BASE.

THEOR. 10. *The product of two or more integer powers of a base is that power of the base whose exponent is the sum of the exponents of the factors.*

Let A be any number, $l, m, n, \cdots$ any positive integers;

then the product $A^l\times A^m\times A^{-n}\cdots$ is $A^{l+m-n}\cdots$

For $\because A^l=A\times A\times\cdots l$ times, $A^m=A\times A\times\cdots m$ times,

$A^{-n}=1/A\times1/A\times\cdots n$ times [df. int. pwr.

$\therefore A^l\times A^m\times A^{-n}=(A\times A\times\cdots\ l+m-n$ times$)\times(A\times1/A)$
$\times(A\times1/A)\times\cdots n$ times, when $l+m\geqslant n$,

and $\because$ $A \times 1/A = 1$,

$\therefore$ $A^l \times A^m \times A^{-n} = A^{l+m-n}$;

and $A^l \times A^m \times A^{-n} = (1/A \times 1/A \times \cdots n - \overline{l+m}$ times)
$(A \times 1/A) \times \cdots l+m$ times when $l+m < n = 1/A^{n-l-m}$
$= A^{l+m-n}$; Q.E.D. [df. neg. pwr.

and so for more than three factors.

COR. *The quotient of two integer powers of the same base is that power of the base whose exponent is the exponent of the dividend less the exponent of the divisor.*

QUESTIONS.

1. By diagram show the square and the cube of 2.
Can the higher powers of 2 be represented by diagrams?

2. Can a concrete number be raised to a power?

3. Of what numbers are the high powers larger than the low powers? smaller? the same?

4. What is the value of $(-2)^4$ and what kind of power is it?
So, of $(+2)^{-3}$, of $(-\frac{1}{3})^3$, of 5^3, of 5^{-3}?

5. What are the products $5^3 \times 5^{-3}$? $(-5)^3 \times (-5)^{-3}$, $a^3 \times a^{-3}$?

6. What power of b is the product $b^m \times b^n \times (1/b)^m$? How many of the factors b are cancelled? how many remain?
So, of the products $b^m \times b^n \times (1/b)^n$; $b^m \times b^n \times (1/b)^m \times (1/b)^n$?

7. What relation has $(1/b)^m$ to b^{-m}?

8. What relation has the quotient $b^m : b^n$ to the products $b^m \times b^{-n}$ and $b^m \times (1/b)^n$?

9. How many times is b a factor of the product $b^l \times b^m$?
How many of these factors can be cancelled by the factors of b^{-n}? how many cannot be so cancelled?

10. Granted that $b^l \times b^m \times b^{-n} = b^{l+m-n}$, what is true of the exponent of the product of integer powers of a number?

11. Divide a^7 by a^3, x^4 by x^{-2}, b^{-2} by b^5, m^{-3} by m^{-2}.

INTEGER POWER OF AN INTEGER POWER.

THEOR. 11. *An integer power of an integer power of a base is that power of the base whose exponent is the product of the two given exponents.*

Let A be any number; m, n any positive integers;
then $(A^m)^n = A^{mn}$.

For $(A^m)^n = A^m \times A^m \times \cdots n$ times
$= A^{m+m+\ldots n \text{ times}} = A^{m\times n}$. Q.E.D. [th. 10.

So, $(A^m)^{-n} = 1/A^m \times 1/A^m \times \cdots n$ times
$= 1/A^{m+m+\ldots n \text{ times}} = 1/A^{mn} = A^{-mn}$.

So, $(A^{-m})^n = A^{-m} \times A^{-m} \times \cdots n$ times
$= A^{-m-m-\ldots n \text{ times}} = A^{-mn}$.

So, $(A^{-m})^{-n} = 1/A^{-m} \times 1/A^{-m} \times \cdots n$ times
$= A^m \times A^m \times \cdots n$ times $= A^{mn}$.

PRODUCT OF LIKE INTEGER POWERS OF DIFFERENT BASES.

THEOR. 12. *The product of like integer powers of two or more bases is the like power of the product of the bases.*

Let A, B, $1/C$ be any numbers and n any positive integer;
then $A^n \times B^n \times 1/C^n = (A \times B/C)^n$.

For $\because A^n = A \times A \times \cdots n$ times, $B^n = B \times B \times \cdots n$ times,
$1/C^n = 1/C \times 1/C \times \cdots n$ times,
$\therefore A^n \times B^n \times 1/C^n = (A \times B \times 1/C) \times (A \times B \times 1/C) \times \cdots$
n times $= (A \times B \times 1/C)^n$. Q.E.D. [ths. 1, 2.

So, $\because A^{-n} = 1/A \times 1/A \times \cdots n$ times, $B^{-n} = 1/B \times 1/B \times \cdots$
n times, $1/C^{-n} = C \times C \times \cdots n$ times,
$\therefore A^{-n} \times B^{-n} \times 1/C^{-n} = (1/A \times 1/B \times C) \times (1/A \times 1/B \times C) \cdots$
n times $= (C/A/B)^n = (A \times B/C)^{-n}$. Q.E.D.

COR. *The quotient of two like integer powers of different bases is the same power of the quotient of the bases.*

EVOLUTION.

Evolution is an operation that is the inverse of involution; i.e., it consists in finding a base that, raised to a given power, is a given number. It is a process of trial and test. The base

is now called the *root*, and the exponent of the power is the *root-index*. The *radical sign*, $\sqrt{\ }$, is placed before the number whose root is sought, and the root-index at the left and above this sign; the root-index 2 need not be written.

E.g., $\sqrt[2]{4}$, or simply $\sqrt{4}$, $= {}^{+}2$, or ${}^{-}2$; $\sqrt[3]{343} = 7$; $\sqrt[3]{{}^{-}343} = {}^{-}7$.

Evolution being the inverse of involution, the following converse theorem follows directly.

THEOR. 13. *An integer root of an integer power of a base is that power of the base whose exponent is the integer quotient of the exponent of the power by that of the root.*

E.g., $\sqrt[m]{A^{mn}} = A^n$.

QUESTIONS.

1. What cases are considered in theor. 11 besides a positive integer power of a positive integer power?

2. Write $(A^{-m})^n$ so that positive exponents only shall be used; then, so as to express the result as a power of A.

3. In the product $A^n \times B^n \times 1/C^n$, how many times is A a factor? B? 1/C? how many factors in the entire product?

Into how many groups of the form $A \times B \times 1/C$ can these factors be divided? Indicate this grouping in the simplest way.

4. Why does $1/C^{-n} = C \times C \times C \cdots n$ times?

5. Write $A^{-n} \times B^{-n} \times (1/C)^{-n}$ with positive exponents.

How many times is the group $1/A \times 1/B \times C$ a factor of this product? what power is it of the fraction $A \times B/C$?

6. Prove that $A^m/B^m = (A/B)^m$.

7. What powers of numbers and of their opposites are the same? not the same?

8. To what element in involution does a root correspond? the root-index? the number whose root is sought?

9. What number has the same effect whether used as a root-index or as an exponent?

10. $\sqrt{4} = {}^{+}2$, $\sqrt{4} = {}^{-}2$: is ${}^{+}2$, or ${}^{-}2$, the square root of ${}^{-}4$?

11. Find the values of $(x^3)^2$, $(y^{-2})^3$, $(b^4)^{-3}$, $(a^{-3})^{-4}$.

12. Find the values of $\sqrt[2]{x^6}$, $\sqrt[3]{y^{-6}}$, $\sqrt[3]{b^{-12}}$, $\sqrt[4]{a^{12}}$.

§ 6. QUESTIONS FOR REVIEW.

Define and illustrate:

1. Quantity; unit; unity; number; integer; fraction.

2. Concrete numbers; equal concrete numbers.

3. Abstract numbers; equal abstract numbers.

4. Product; multiplication; division; quotient; the product of two abstract numbers.

5. Positive numbers; negative numbers.

6. Sum; addition; subtraction; remainder; the sum of two abstract numbers.

7. Reciprocals; opposites; a number larger than another.

8. Power; involution; evolution; root.

State and prove:

9. The associative principle of multiplication.

10. The commutative principle of multiplication, four cases.

11. The associative and commutative principles of addition.

12. The distributive principle of multiplication.

13. The principle on which the theory of division rests.

14. The principle on which the theory of subtraction rests.

15. The principle by which the product of two integer powers of the same base is found.

How does this principle apply to their quotient?

16. The principle by which an integer power of an integer power is found.

How does this principle apply to the extraction of roots?

17. The principle by which the product of like integer powers of different bases is found.

How does this principle apply to their quotient?

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18. What two offices has an abstract negative multiplier?

19. How can a series of additions and subtractions be made? a series of multiplications and divisions? with what caution?

20. What is the product of the reciprocals of two or more numbers? the sum of their opposites?

21. What operation is the inverse of multiplication? of addition? of involution? What inverses has the multiplication $\$5 \times 4$? the involution 4^3?

22. Can two concrete numbers be added together? with what caution? Can one such number be subtracted from another?

A man may walk a mile, then take another step: can a mile and a yard be added together?

23. Can two concrete numbers be multiplied together? Can one such number be divided by another? with what caution?

24. Can a concrete number be raised to a power? Can a root be taken of such a number?

25. What is the meaning of a concrete fraction? of an abstract fraction? Are both terms of a concrete fraction concrete?

26. What effect has it upon a fraction if both terms be multiplied by the same number? if both be divided by the same number? if the same number be added to both terms? if both terms be raised to the same power? if like roots be taken?

27. Group the factors below in such a way as to make the multiplication easiest; hence explain cancellation in the multiplication of fractions: $3/2 \cdot 7/11 \cdot 25/51 \cdot 2/7 \cdot 11/5 \cdot 17/5$.

28. Is involution distributive as to multiplication? is evolution? Are these two statements true equations?

$$(12 \times 6)^2 = 12^2 \times 6^2, \quad \sqrt[3]{(27 \times 8)} = \sqrt[3]{27} \times \sqrt[3]{8}.$$

29. Is involution distributive as to addition? is evolution? Are these two statements true equations?

$$(12 + 6)^2 = 12^2 + 6^2, \quad \sqrt[3]{(27 \pm 8)} = \sqrt[3]{27} \pm \sqrt[3]{8}.$$

30. Suggest questions that could not possibly be answered by a negative number; not, by a fraction.

31. Show that multiplication may be defined as the process of doing to one of two numbers that which, when done to a unit, produces the other.

32. Why is it that, instead of dividing by a composite number, one may divide by one of its factors, that quotient by another factor, and so on, till all the factors are used?

33. Why does $\sqrt{(108 \times 27 \times 250 \times 490)}$ equal $6 \times 9 \times 5 \times 70$?

II. THE PRIMARY OPERATIONS OF ALGEBRA.

In principle, the operations of algebra differ not at all from the like operations of arithmetic; the only differences arise from the differences between the forms of algebraic expressions and the simpler forms common in arithmetic.

E.g., the sum of 4 and 5 is 9, and their product is 20;

but the sum and product of a and b can only be expressed by writing $a+b$ and $a \times b$, $a \cdot b$, or ab,

which mean that an addition and a multiplication are intended, and that they will be effected when the values of a and b are made known.

§ 1. ALGEBRAIC EXPRESSIONS.

An *algebraic expression* is a number or combination of numbers written in algebraic form. It is called an *expression* or a *number*, according as the thought is of the symbol or of the value that the symbol represents.

Unless it be a single letter or numeral, an expression is made up of simpler expressions affected or combined by signs of operation.

The parts of an expression that are joined by the signs $+$ or $-$ are *terms*.

An expression of one term only is a *monomial*, of two terms a *binomial*, of three terms a *trinomial*, of four terms a *quadrinomial*, of two or more terms a *polynomial*.

An expression is *numerical* if the numbers be expressed wholly by numerals, *literal* if wholly or in part by letters; *finite* if the number of operations implied be limited, *infinite* if unlimited.

An algebraic expression is *entire* if it be free from divisors and roots, *fractional* if not free from divisors.

When the terms of an expression are so related to each other that each successive term is derivable by some fixed law from the previous terms, the expression is a *series*.

E.g., $1+x+x^2+x^3+\cdots+x^r$ is a *finite series*, if r be any given integer, arranged according to *rising powers of* x;

but $1+x+x^2+x^3+\cdots+x^r+\cdots$ is an *infinite series*.

QUESTIONS.

1. Is a/b an algebraic expression or a number?

If a be the entire cost of b books, what is a/b?

2. What name is given to the parts a, b, of the expression a/b? of the expressions $a-b$, $a+b$, $a\cdot b$, a^b, $\sqrt[b]{a}$?

3. What three pairs of contrasting names are applied to algebraic expressions? define and illustrate them.

4. In the series $1+x+x^2+\cdots+x^r$, how is the third term got from the second? the fourth from the third?

If r be 7, how many terms are in this series?

How many operations are performed? what are they?

Is the series finite or infinite?

5. In the series $1+x+x^2+\cdots+x^r+\cdots$, how many terms are there after x^2? What is the twelfth term of this series? the twentieth? the nth?

6. If $a=4$, $b=1$, $c=2$, $d=9$, $x=5$, $y=8$, find the values of

$$\sqrt{8ac},\quad 7\sqrt{5dx},\quad a^2-2b^4-3abc,\quad \sqrt{y^a},\quad \sqrt{b^d},\quad (2d-5c)^3.$$

7. If $a=1$, $b=-3$, $c=5$, find the value of

$$\frac{a^2b^2+1}{a^2+b^2}-\frac{1-a^2c^2}{a^2-c^2}+\frac{2b^2-4ac}{b^2-c^2}-\frac{a^2+2ab+b^2}{b^2-2bc+c^2}.$$

8. If $a=25$, $b=9$, $c=-4$, $d=-1$, find the value of

$$\sqrt{-bc}+3\sqrt{acd}-4\sqrt{-b^2d}+\sqrt{-c^2d}.$$

9. If $a=0$, $b=-2$, $c=4$, $d=-6$, find the value of

$$3\sqrt[3]{(2b^2-a)}+2\sqrt[3]{(b^2+c^2+7)}-\sqrt[3]{[2(b+c)^2+(d+b)^2+bc]}.$$

10. If $x=1/2$, how much will the sum of the first three terms of the series $1+x+x^2+x^3+\cdots$ lack of 2?

So, the first four terms? five terms? ten terms? n terms?

SYMMETRY.

As to any of its letters, an expression is *symmetric* when its value is unchanged however those letters exchange places.

E.g., the expressions $x \times y \times z$ and $x+y+z$ are symmetric as to x, y, z, or as to any two of them.

So, $w+x-y-z$ is symmetric as to w, x, and as to y, z; but not as to w, y, as to w, z, as to x, y, or as to x, z.

DEGREE—HOMOGENEOUS TERMS.

The sum of the exponents in a monomial is its *degree*. The degree of a polynomial is that of the term whose degree is highest of all.

A polynomial made up of terms all of the same degree is *homogeneous*. Expressions having the same degree are *homogeneous with each other*.

E.g., $a^3 + 3a^2b + 3ab^2 + b^3$ is homogeneous and of the third degree.

So, $a^n, a^{n-1}b, a^{n-2}b^2, \cdots a^{n-r}b^r, \cdots ab^{n-1}, b^n$ are homogeneous with each other and of the nth degree.

And ax^2, b^2xy, c^3y^2 are of the second degree, and homogeneous with each other as to x and y,

but of the third, fourth, fifth degrees, and not homogeneous, as to all the letters.

COEFFICIENTS—LIKE AND UNLIKE TERMS.

If an expression have a numerical factor, then usually the numeral alone, together with the sign of the number, + or −, or the numeral and one or more of the letters that follow it, is called the *coefficient*.

E.g., in $7abc$, 7 is the coefficient of abc, $7a$ of bc, $7ab$ of c.

Terms that differ only in their coefficients are *like* (similar) terms; other terms are *unlike*.

E.g., $5ax$, $7ax$ are like; but $5ax$, $7by$ are unlike.

So, $5ax$, $7bx$ are like if $5a$, $7b$ be counted as the coefficients of x; but unlike if 5, 7 be coefficients of ax, bx.

QUESTIONS.

1. If $x=2$, $y=3$, $z=4$, $w=6$, is the expression $x\times y\times z$ the same as $2\times 3\times 4$? as $3\times 2\times 4$? $4\times 2\times 3$? $2\times 4\times 3$? $3\times 4\times 2$? $4\times 3\times 2$?

So, is $w+x-y-z$ the same as $6+2-3-4$? $4+6-3-2$? $2+6-4-3$? $6+3-2-4$?

2. In the expression $a\times b\times c\times d$, what letters may be interchanged without changing the value?

So, in the expression $a+b-c+d$? in ab/cd? in $ab+cd$?

3. What is the degree of a? of a^2? of ab?

4. Of the trinomial $a^2-abc+c$, which term is of the highest degree?

What is the degree of this expression? is it homogeneous?

Supply the terms that are wanting so as to complete the homogeneous series:

5. $a^6+a^5b+a^4b^2+\cdots+ab^5+b^6$.

6. $a^7+a^6b+a^5b^2+\cdots+ab^6+b^7$.

7. $a^6-a^5b+a^4b^2-\cdots-ab^5+b^6$.

8. $a^7-a^6b+a^5b^2-\cdots+ab^6-b^7$.

9. $a^n+a^{n-1}b+a^{n-2}b^2+\cdots+ab^{n-1}+b^n$.

10. $a^n-a^{n-1}b+a^{n-2}b^2-\cdots-ab^{n-1}+b^n$.

11. $a^n-a^{n-1}b+a^{n-2}b^2-\cdots+ab^{n-1}-b^n$.

12. If n be even will the series $a^n-a^{n-1}b+\cdots$ end with $-b^n$ or with $+b^n$? if n be odd?

13. Supply the terms that are wanting and complete the symmetric series $a^3+a^2b+a^2c+\cdots+abc+\cdots+c^3$.

14. So, $a(x^3+y^3+z^3)+b(x^2y+x^2z+\cdots)+cxyz$.

15. If an expression be symmetric as to every pair of letters involved, it is symmetric as to all the letters, and conversely.

16. In the product $\frac{1}{2}a^2bc^3$ what is the coefficient of bc^3? of c^3?

17. Are $3ax$, $5a^2x$, $-\frac{1}{2}bx$, $\frac{3}{4}b^2x$, like terms?

Can they be so taken as to be made like terms?

18. As an operator, what process does a coefficient indicate?

§ 2. ADDITION AND SUBTRACTION.

Prob. 1. To add two or more numbers.

(a) *The numbers like: to the common factor, prefix the sum of the coefficients.* [I th. 7.

E.g., 10 ft. down + 20 ft. up + 60 ft. up = 70 ft. up,
10 ft. up + 20 ft. down + 60 ft. down = 70 ft. down.

So, $10x - 15x + 20x - 25x + 30x = 60x - 40x = 20x$,
$10ay + 20by - 30cy = (10a + 20b - 30c)y$.

(b) *The numbers unlike: write the numbers in succession, with their signs, in any convenient order.* [I th. 5.

E.g., $19xyz - 29mn + 39a - 49$ is irreducible.

So, $10ay + 20by - 30cy$ is usually not reduced, but may be written $(10a + 20b - 30c)y$, as above.

(c) *Some numbers like and some unlike: unite into one sum each set of like numbers;*

write these partial sums, together with the remaining terms, in any convenient order.

E.g., $(a^3 + 3a^2b + 3ab^2 + b^3) + (a^3 - 3a^2c + 3ac^2 - c^3)$
$= 2a^3 + 3a^2(b - c) + 3a(b^2 + c^2) + (b^3 - c^3)$.

SUBTRACTION.

Prob. 2. To subtract one number from another.

By trial, or memory, find what number added to the subtrahend gives the minuend. [df. sub., p. 28.

Or, *to the minuend add the opposite of the subtrahend.* [I th. 9.

In general, the first rule is best when the numbers are like, and the second, when they are unlike.

E.g., $7a - 4a = 3a$, $7a - {}^{-}3a = 10a$.

So, ${}^{-}7a - 3a = {}^{-}10a$, ${}^{-}7a - {}^{-}3a = {}^{-}4a$.

So, $7a - {}^{-}4b = 7a + 4b$, $7a - {}^{+}4b = 7a - 4b$.

So, $[2a^3 + 3a^2(b - c) + 3a(b^2 + c^2) + (b^3 - c^3)] - [a^3 - 3a^2c + 3ac^2 - c^3]$
$= a^3 + 3a^2b + 3ab^2 + b^3$.

QUESTIONS.

Add

1. $7x$, $15x$, ^-3x, ^-8x, $2x$, $12x$, ^-15x, $27x$, ^-31x.

2. $3a^2y$, $18a^2y$, $^-25a^2y$, $^-42a^2y$, $33a^2y$, $25a^2y$.

3. $7ax^2$, $^-4a^2x$, $5a^2x$, $9a^2x$, $^-15ax^2$, $^-26a^2x$.

4. $7a+5c-3x^2y$, $5x^2y-8b-4a$, $3b+4x^2y+2c$.

5. $8mn^2+3x^2y^3+5a$, $7x^2y^3+3mn^2-7a$, $2mn^2-17x^2y^3+3a$.

6. $(a-2p)x^3$, $(q-b)x^2$, $(3c-2r)x$, $(3p-a)x^3$, $-(b+q)x^2$, $-(p-a)x$, [arrange the sum to rising powers of x.

7. $a^6-a^5b+a^4b^2-\cdots-ab^5$, $a^5b-a^4b^2+a^3b^3-\cdots-b^6$.

8. $a^7-a^6b+a^5b^2-\cdots+ab^6$, $a^6b-a^5b^2+a^4b^3-\cdots+b^7$.

9. $a^n-a^{n-1}b+a^{n-2}b^2-\cdots\pm ab^{n-1}$, $a^{n-1}b-a^{n-2}b^2+\cdots\pm b^n$.

10. From $7x$ subtract $3x$, $5x$, $7x$, $9x$, $11x$, in turn, and add the five remainders.

11. So, from $13ay^2$ subtract $5ay^2$, $3ay^2$, ay^2, $-ay^2$, $-3ay^2$.

Find the value of

12. $(a+b+c-2+5x)-(a-b-9+13x)+(a+b-c+8x)$.

13. $(7+5b)-[(5ax+3b-2)-(4m+3ax-4b)]$.

14. $(8xy+14x^2y^3z-8x^3yz^4)-(3xy-8x^3y^2z-7x^3yz^4)$.

Free from brackets and reduce to the simplest form

(*a*) removing first the inner brackets, and going outwards;
(*b*) removing first the outer brackets, and going inwards;
(*c*) freeing all terms of a kind, from all the brackets:

15. $a-[b-(e-d)]$. 16. $a-\{a+b-[a+b-c-(a-b+c)]\}$.

17. $-\{(1+2x+9x^2)+[(3+2x-x^2)-(-3+3x-2x^2)]\}$.

Introduce brackets, taking the terms two together in their present order, and having

(*a*) each bracket preceded by a plus sign;
(*b*) each bracket preceded by a minus sign;
(*c*) the first term in each bracket positive:

18. $-3c+4d-2e+2f+2a-5b$.

19. $a+b+c-a-b+c+a-b-c-a+b-c$.

20. $abc-abd+abe-acd+ace+ade-bcd+bce-bde+cde$.

§ 3. MULTIPLICATION.

Prob. 3. To multiply one number by another.

(*a*) *A monomial by a monomial: to the product of the numerical coefficients, annex the several letters, each taken as many times as it appears in both factors together;* [I ths. 1, 2, 10.

mark the product positive if the factors be both positive or both negative, and negative if one factor be positive and the other negative. [df. neg. oper., p. 20.

E.g., ${}^{+}9ab^{-3}\times{}^{+}7a^2={}^{+}63a^3b^{-3}$, $\quad {}^{-}5xz^{-3}\times{}^{+}7x^3z^{-2}={}^{-}35x^4z^{-5}$,

${}^{+}9a^{-3}b^3\times{}^{-}7a^2={}^{-}63a^{-1}b^3$, $\quad {}^{-}5xz^{-3}\times{}^{-}7x^{-4}z^3={}^{+}35x^{-3}$.

(*b*) *A polynomial by a monomial: multiply each term of the multiplicand by the multiplier;*

add the partial products. [I th. 7.

E.g., $(3xy^2-\frac{3}{5}x^{-5}z)\times{}^{-}\frac{5}{8}xy^{-3}z^5=-\frac{15}{8}x^2y^{-1}z^5+\frac{3}{8}x^{-4}y^{-3}z^6$.

(*c*) *A polynomial by a polynomial: multiply each term of the multiplicand by each term of the multiplier;*

add the partial products. [I th. 7 cr.

E.g., $(a^2-ab+b^2)\times(a+b)=a^3-a^2b+ab^2+a^2b-ab^2+b^3$
$=a^3+b^3$.

The work takes this form:

$$\begin{array}{l} a^2-ab+b^2 \\ a+b \\ \hline a^3-a^2b+ab^2 \\ \quad +a^2b-ab^2+b^3 \\ \hline a^3 \qquad\qquad +b^3 \end{array}$$

So, to multiply $ax^2+2bxy+cy^2+2dxz+2eyz+fz^2$ by $mx+ny+pz$: write

$$\begin{array}{l} ax^2+2bxy+cy^2+2dxz+2eyz+fz^2 \\ mx \qquad\quad +ny \qquad\qquad\quad +pz. \end{array}$$

$$\begin{array}{ll|l|ll|l|l|ll|l|ll} \hline amx^3+2bm & & x^2y+cm & xy^2 & +2dm & x^2z+2em & xyz & & +fm & xz^2 & & \\ & +an & +2bn & +cny^3 & & +2dn & +2cn & y^2z & & +fn & yz^2 & \\ & & & & +ap & +2bp & +cp & & +2dp & +2ep & & +fpz^3. \end{array}$$

Checks. The work is tested by division [prob. 4], and often by the principles of note 1 below.

QUESTIONS.

1. In repetition by a positive multiplier, what relation has the sign of the product to that of the multiplicand? in partition?

2. Of what two factors is a negative multiplier composed? Does the tensor change the sign of the multiplicand? does the versor?

What is the sign of the product of a positive number by a negative number? of a negative number by a negative number?

Multiply

3. $(a+b)\times(c-d)$. 4. $(2ab-c^2+3x)\times(3ab-c+x)$.

5. $(3a^3-2a^2b-ab^2+4b^3)\times(a^2+2ab-b^2)$.

6. $(x^2-2xy+y^2)\times(x^2+y^2)\times(x^2+2xy+y^2)\times(x^2-y^2)$.

7. $5(x^4+x^3y+x^2y^2+xy^3+y^4)\times(x-y)$.

8. $(4c^2d-7cd^2+3d^3)\times(5c^3+4c^2d-9cd^2)$.

9. $(x^{2n}+x^ny^n+y^{2n})\times(x^n-y^n)\times(x^{2n}-x^ny^n+y^{2n})\times(x^n+y^n)$.

10. $(b^2-bc+c^2)\times(b^2+bc+c^2)\times(b+c)\times(b-c)$.

11. $(5xy-3xz+2yz)\times(ax-2by+3cz)$.

12. $(3a-x+2)\times(a-4x-1)$. 13. $(x^3-ax^2+7y^2)\times(x-ay)$.

14. $(3a^2x-ax+x)\times(2a-y)\times(3a+z)$.

15. $(4a^{-3}-2x^{-2}+x^{-1}-6)\times(3x^2-a^3)$.

16. $(3a^2+5ab-4ac+8b^2-c^2)\times(a-2bc+3c^2)$.

17. $[5x^2y-(b-3c)]\times(b-1)\times(x+c)$.

18. $(2xy+x^2-3y^2)\times(3x^2-xy+2y^2)$.

19. $(a^4-2a^3+3a^2-2a+1)\times(a^4+2a^3+3a^2+2a+1)$.

20. $(5x^3+4x^2-3-7x^{-1}+10x^{-3})\times(x^3+x-2)$.

21. $(17a^3bx^{-1}+12ab^{-2}x^2-9bc^2)\times(5b^2x+6a^{-1}b^{-1})$.

22. $(3ay^2+4a^2y-6-10b^{-1}y^{-1}-5y^{-2})\times(y^2-\frac{1}{2}by)$.

23. $(\frac{1}{2}x^3-x^2+\frac{3}{2}x-2)\times(2x^3+\frac{3}{2}x^2+x+\frac{1}{2})$.

24. $(x^2+2x-3)\times(x^2-x+1)$. 25. $[x^2-(b+c)x+bc]\times(x-a)$.

26. $[x^5+a^5-ax(x^3+a^3)]\times[x^3+a^3-ax(x+a)]$.

27. $(x^3-3x^2+3x-1)\times(x^2+2x+1)\times(x+1)$.

FORM OF PRODUCT.

NOTE 1. Certain general principles are manifest:

1. *The form of a product is independent of the values of the letters that enter into it.*

E.g., $(a+b)\times(a-b)=a^2-b^2$, whatever be the values of a, b.

2. *If each factor be symmetric as to two or more letters, the product is also symmetric as to the same letters.*

E.g., $(a^2+2ab+b^2)\times(a+b)=a^3+3a^2b+3ab^2+b^3$.

3. *If any values be given to the letters, the value of the product equals the product of the values of the factors.*

E.g., if $a=5$, $b=2$, then $a^2+2ab+b^2=25+20+4=49$, [above.

$$a+b=5+2=7,\quad a^3+3a^2+3ab^2+b^3=49\cdot 7=343.$$

4. *The sum of the coefficients of a product is the continued product of the sum of the coefficients of the first factor, by the sum of the coefficients of the second factor, and so on.*

E.g., $(4a+3b)\times(5a-2b)\times(a-3b)=20a^3-53a^2b-27ab^2+18b^3$.

$$(4+3)\times(5-2)\times(1-3)=20-53-27+18=-42.$$

5. *The degree of the highest term of a product, as to any letter or letters, is the sum of the degrees of the highest terms of the factors, as to the same letter or letters; and so of the lowest term.*

E.g., in $(a^2+2ab+b^2)\times(a+b)=a^3+3a^2b+3ab^2+b^3$ the degree of the highest terms of the factors as to a are 2, 1, and of the highest term of the product, it is 3.

So, the degrees of the lowest terms as to a are 0, 0, 0.

6. *If each factor be homogeneous as to any letter or letters, the product is homogeneous as to the same letter or letters.*

E.g., in the example just above, the factors and the product are all homogeneous as to the two letters a, b.

7. *The whole number of terms in any product, before reduction, is the continued product of the number of terms in the several factors; and the product of two or more polynomials can never be reduced to less than two terms, that of highest degree and that of lowest degree as to any letter or letters.*

QUESTIONS.

1. For how many different values of a, b, is the statement $(a-b)^2 = a^2 - 2ab + b^2$ true?

Is the value of $a^2 - 2ab + b^2$ the same, whatever values be given to a, b? Why, then, is the statement always true?

2. In the product $(a^2 + 2ab + b^2) \times (a + b)$ if every b be replaced by an a, and every a by a b, is the multiplicand changed? the multiplier? the product?

As to what letters are these three expressions symmetric?

3. As to what letters is the product

$(a^2 - abc + b^2) \times (a + b - c)$ symmetric?

Replace a by 1, b by 3, c by 5, and test the product.

To make the test doubly sure, replace a, b, c by other numerals and test it again.

Is the test good whatever values be given to a, b, c?

Is it a perfect test?

What are the most convenient values?

4. Find the product $(x-2) \times (2x-1)$ and test it by showing that the sum of the coefficients of the product is the product of the sums of the coefficients of the factors.

5. So, $(x-2) \times (2x-1) \times (x-4) \times (4x-1) \times (x-6) \times (6x-1)$.

How many terms has this product before reduction? after reduction?

6. Show that the fourth principle is only another way of stating the third when the value 1 is given to each letter.

7. In the product $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$, what term contains a with exponent 0? what is the value of a^0?

8. Of what degree, as to x, is the product $(x^3 + y) \times (x^2 + y)$? as to y? as to x and y?

So, the product $(x^3y + y^2) \times (x^2y^2 + y) \times (xy^3 + y^0)$?

9. Of what degree is the product of

$a^m + a^{m-1}b + a^{m-2}b^2 + \cdots + ab^{m-1} + b^m$ by $a^n + a^{n-1}b + \cdots + b^n$.

Is this product homogeneous? symmetric?

10. Multiply $4b^3 - 3b^2c - 3bc^2 + 4c^3$ by $2b + 2c$, and show the application of the seven laws just pointed out.

ARRANGEMENT.

Note 2. The work is often shortened by arranging the terms of both factors, and of the product, as to the powers of some one letter and grouping together like partial products.

E.g., $(a^3+3ab^2+3a^2b+b^3)\times(b^2+2ab+a^2)$ is written

$$\begin{array}{llllllllll}
\multicolumn{10}{l}{a^3+3a^2b+3ab^2+b^3} \\
\multicolumn{10}{l}{a^2+2ab\ +b^2} \\
\hline
a^5 & +3 & a^4b & +3 & a^3b^2 & +1 & a^2b^3 & & & \\
 & +2 & & +6 & & +6 & & +2 & ab^4 & \\
 & & & +1 & & +3 & & +3 & & +b^5 \\
\hline
a^5 & +5 & a^4b & +10 & a^3b^2 & +10 & a^2b^3 & +5 & ab^4 & +b^5.
\end{array}$$

CROSS MULTIPLICATION.

Note 3. The work is often shortened by picking out and adding mentally all like partial products, and writing their sum only. In ex. nt. 2 the computer says and writes

$a^3\times a^2$ is	a^5
$3a^2b\times a^2$ is $3a^4b$, $a^3\times 2ab$ is $2a^4b$, whose sum is	$5a^4b$
$3ab^2\times a^2$ is $3a^3b^2$, $3a^2b\times 2ab$ is $6a^3b^2$, $a^3\times b^2$ is a^3b^2, whose sum is	$10a^3b^2$
$b^3\times a^2$ is a^2b^3, $3ab^2\times 2ab$ is $6a^2b^3$, $3a^2b\times b^2$ is $3a^2b^3$, whose sum is	$10a^2b^3$
$b^3\times 2ab$ is $2ab^4$, $3ab^2\times b^2$ is $3ab^4$, whose sum is	$5ab^4$
$b^3\times b^2$ is	b^5

and the product is $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

So, to multiply 384 by 287, product 110208, 384

i.e., $3\cdot10^2+8\cdot10+4$ by $2\cdot10^2+8\cdot10+7$, 287

the computer says and writes

28;	8
2; 56, 58; 32, 90;	0
9; 21, 30; 64, 94; 8, 102;	2
10; 24, 34; 16, 50;	0
5; 6, 11;	11

wherein the numbers 28, 56, 32, 21, 64, 8, 24, 16, 6, are the pr'd'ts $4\cdot7, 8\cdot7, 4\cdot8, 3\cdot7, 8\cdot8, 4\cdot2, 3\cdot8, 8\cdot2, 3\cdot2$.

QUESTIONS.

Multiply, and check the work:

1. $(x^3-2x^2-3x+1)\times(2x^2-3x+4)$.

2. $(1+x^3-3x-2x^2)\times(4+2x^2-3x)$.

3. $(x^2+y^2+z^2+xy+yz-zx)\times(x-y+z)$.

4. $(x^2-zx+y^2+yz+z^2+xy)\times(z+x-y)$.

5. $(x^2+ax-b^2)\times(x^2+bx-a^2)\times(x-\overline{a+b})$.

6. $(b^2-x^2-ax)\times(bx-a^2+x^2)\times(a+b-x)$.

7. $[(a-1)x^3+(a-1)^2x^2+(a-1)^3x]\times[(a+1)x+(a+1)^3x^{-1}]$.

8. $(x^2-ax+a^2)\times(a+x)$; $(ax-x^2-a^2)\times(x+a)$.

9. $(a-b+c)\times(b-c+a)\times(c-a+b)\times(a+b+c)$.

10. $(1+x+x^2+x^3+x^4+x^5)\times(1-x+x^2-x^3+x^4-x^5)$.

11. $(5x^5+4x^4+3x^3+2x^2+x)\times(5x^5-4x^4+3x^3-2x^2+x)$.

12. $(a^m+3b^n-2c^p)\times(a^{-m}-3b^{-n}+2c^{-p})$.

13. $(a^m-2c^p+3b^n)\times(3b^{-n}+a^{-m}+2c^{-p})$.

14. $\left.\overline{\begin{matrix}x^2+a\\+b\end{matrix}}\right|\overline{x+ab}\times\left.\overline{\begin{matrix}x^2+c\\+d\end{matrix}}\right|\overline{x+cd}$; $\left.\overline{\begin{matrix}x^2+a\\+b\end{matrix}}\right|\overline{x+ab}\times\left.\overline{\begin{matrix}x^2-c\\-d\end{matrix}}\right|\overline{x+cd}$.

15. $\left.\overline{\begin{matrix}x^2-a\\-b\end{matrix}}\right|\overline{x+ab}\times\left.\overline{\begin{matrix}x^2+c\\+d\end{matrix}}\right|\overline{x+cd}$; $\left.\overline{\begin{matrix}x^2-a\\-b\end{matrix}}\right|\overline{x+ab}\times\left.\overline{\begin{matrix}x^2-c\\-d\end{matrix}}\right|\overline{x+cd}$.

16. $(x+a)\times(x+b)\times(x+c)\times(x+d)$. [at one operation.

17. $(x+a)\times(x+b)\times(x+c)$; $(x-a)\times(x-b)\times(x-c)$.

18. $(x-a)\times(x+b)\times(x-c)$; $(x-a)\times(x-b)\times(x+c)$.

19. 13×15; 35×79; 234×432; 23.4×4.32; 135.7×12.34.

20. 18^2; 37^2; 109^2; 163^2; 725^2; 1881^2; 70.23^2; 0.205^2.

21. Given $a+b+c=p_1$, $bc+ca+ab=p_2$, $abc=p_3$;
show that $a^2+b^2+c^2=p_1^2-2p_2$,
and that $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2=p_1p_2-3p_3$.

22. So, express $a^3+b^3+c^3$ in terms of p_1, p_2, p_3, using the identity

$$a^3+b^3+c^3-3abc=(a^2+b^2+c^2-bc-ca-ab)(a+b+c).$$

TYPE-FORMS.

Note 4. The work is often shortened by the use of certain simple type-forms, which the pupil may prove by actual multiplication and then memorize. He may also translate them into words and read them as theorems.

E.g., Form [2] may be read: *the product of the sum and the difference of two numbers is the difference of their squares.*

1] $(x+a)\times(x+b)=x^2+(a+b)x+ab.$

E.g., $(x+6)\times(x-2)=x^2+(6-2)x-12=x^2+4x-12.$

So, $(a-4)\times(a-3)=a^2-7a+12.$

2] $(a+b)\times(a-b)=a^2-b^2.$

E.g., $(a+5)\times(a-5)=a^2-25.$

So, $(a+\overline{b-3})\times(a-\overline{b-3})=a^2-\overline{b-3}^2=a^2-b^2+6b-9.$

3] $(a+b)^2=a^2+2ab+b^2.$

E.g., $(2y+7)^2=\overline{2y}^2+2\cdot 2y\cdot 7+7^2=4y^2+28y+49.$

So, $(2x+3a)^2=4x^2+12ax+9a^2.$

4] $(a-b)^2=a^2-2ab+b^2.$

E.g., $(3z-8)^2=9z^2-2\cdot 3z\cdot 8+8^2=9z^2-48z+64.$

So, $(\overline{a+b}-c)^2=a^2+2ab+b^2-2ac-2bc+c^2.$

5] $(a+b+c+\cdots)^2=a^2+b^2+c^2+\cdots$
$+2(ab+ac+\cdots+bc+\cdots).$

E.g., $(a+b-c-d)^2=a^2+b^2+c^2+d^2$
$+2(ab-ac-ad-bc-bd+cd).$

6] $(a-b)\times(a^{n-1}+a^{n-2}b+a^{n-3}b^2+a^{n-4}b^3+\cdots+ab^{n-2}+b^{n-1})$
$=a^n-b^n,$ if n be any integer.

E.g., $(x-2)\times(x^2+2x+4)=x^3-2^3=x^3-8.$

So, $(2c-a^2)\times(8c^3+4a^2c^2+2a^4c+a^6)=16c^4-a^8.$

7] $(a+b)\times(a^{n-1}-a^{n-2}b+a^{n-3}b^2-a^{n-4}b^3+\cdots+ab^{n-2}-b^{n-1})$
$=a^n-b^n,$ if n be any even integer.

E.g., $(x+2)\times(x^5-2x^4+4x^3-8x^2+16x-32)=x^6-64.$

8] $(a+b)\times(a^{n-1}-a^{n-2}b+a^{n-3}b^2-a^{n-4}b^3+\cdots-a\cdot b^{n-2}+b^{n-1}$
$=a^n+b^n,$ if n be any odd integer.

E.g., $(m^2+3l)\times(m^8-3m^6l+9m^4l^2-27m^2l^3+81l^4)=m^{10}+243l^5.$

QUESTIONS.

Name the type-form that applies, find the product, and check the work by the principles of note 1.

1. $(x+2)\times(x+3)$. 2. $(x+2)\times(x-3)$. 3. $(x-2)\times(x+3)$.
4. $(x-2)\times(x-3)$. 5. $(y+a)\times(y+b)$. 6. $(y-a)\times(y-b)$.
7. $(y-a)\times(y+b)$. 8. $(y+a)\times(y-b)$. 9. $(a+2)(a+1)$.
10. $(x-5)\times(x-6)$. 11. $(a-4)\times(a-3)$. 12. $(5+x)\times(5-y)$.
13. $(x+\overline{a+b})\times(x+\overline{c+d})$. 14. $(x-\overline{a+b})\times(x-\overline{c+d})$.
15. $(3a^2x^3+5b^3c^2)\times(3a^2x^3+5b^2c^3)$.
16. $(a+9)\times(a-9)$. 17. $(3-a)\times(3+a)$. 18. $(2+x)\times(2-x)$.
19. $(k+1)\times(k-1)$. 20. $(x^2+2)\times(x^2-2)$. 21. $(x^3+7)\times(x^3-7)$.
22. $(50-5)(50+5)$; 45×55; 78×82; 47×53; 99×101.
23. $(ax^2+y^3)\times(ax^2-y^3)$. 24. $(2x^3+3y^2z)\times(2x^3-3y^2z)$.
25. $(x-a)\times(x+a)$; $(x^2-a^2)\times(x^2+a^2)$; $\cdots$; $(x^n-a^n)\times(x^n+a^n)$.
26. $(1-x)\times(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)\cdots(1+x^{128})$.
27. Show that the rules that apply to form 1 apply also to forms 3, 4. How does the square of the sum of two numbers differ from the square of their difference?

Find the values:

28. $(x+6a)^2$. 29. $(3x-2y)^2$. 30. $(a^3-1/2)^2$. 31. $(ab-4)^2$.
32. $(x+3y)^2$. 33. $(x-3y)^2$. 34. $(x\pm3)^2$. 35. $(2x^2\pm3y^2)^2$.
36. $(a+\overline{b-c})^2$. 37. $(a-\overline{b+c})^2$. 38. $(\overline{a+b}\pm\overline{c-d})^2$.
39. $(100-1)^2$; 99^2; $(61)^2$; 28^2; 73^2; 807^2; 8.07^2; $.0807^2$.
40. In the square of a polynomial of five terms, how many terms are perfect squares? How are the other terms formed?

Find the values:

41. $(x+y+z)^2$. 42. $(2x+3y-4z)^2$. 43. $(xy+yz+zx)^2$.

Find the products:

44. $(a^2+ax+x^2)\times(a-x)\times(a^2-ax+x^2)\times(a+x)$.
45. $(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})\times(x-y)$.
46. $(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots\pm xy^{n-2}\mp y^{n-1})\times(x+y)$.
47. $(p+pr+pr^2+pr^3+\cdots+pr^{n-1})\times(1-r)$.

DETACHED COEFFICIENTS.

NOTE 5. If both multiplicand and multiplier be such that, when their coefficients are detached, the remaining factors of pairs of successive terms have a constant ratio, the work is shortened by writing down the coefficients only.

Take the terms of both factors in such order that when the coefficients are detached the parts left, taken two and two in order, have a constant ratio;
in place of the given polynomials, write the two groups of coefficients, with 0 *for the coefficient of any absent term;*
multiply the coefficients as polynomials; add those partial products that pertain to like terms of the final product;
in the final product restore the suppressed letters.

E.g., $(a^3+3a^2b+3ab^2+b^3)\times(a^2+2ab+b^2)$ [ratio $b:a$.

$$
\begin{array}{llllll}
\text{gives } 1+3+ & 3+ & 1 & & & \\
1+2+ & 1 & & & & \\
\hline
1+3+ & 3+ & 1 & & & \\
+2+ & 6+ & 6+2 & & & \\
 & +1+ & 3+3+1 & & & \\
\hline
1+5+ & 10+ & 10+5+1; & & &
\end{array}
$$

and the product is $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

CHECK: $1+3+3+1=8$, $1+2+1=4$,
$8\cdot 4=32$, $1+5+10+10+5+1=32$.

This method is a familiar one in arithmetic; the ratio is 10.

E.g., $1089\times 237=258093$, or

$$
\begin{array}{r}
237 \\
\hline
7623 \\
3267 \\
2178 \\
\hline
258093
\end{array}
\qquad
\begin{array}{r}
1th+0h+8t+9u \\
2h+3t+7u \\
\hline
7th+6h+2t+3u \\
3tth+2th+6h+7t \\
2hth+1tth+7th+8h \\
\hline
2hth+5tth+8th+0h+9t+3u.
\end{array}
$$

The first form is a case of detached coefficients, wherein the denominations and the relations of the several figures are shown by their positions, as in the last form they are shown by words and signs.

QUESTIONS.

Multiply and check the work:

1. $(2x+3)\times(3x-4)$. 2. $(x^2+3x+2)\times(x^2-3x+2)$.

3. $(3y-5)\times(2y+7)\times(2-4y^2)\times(1+2y^2)$.

4. $(x^3+3x^2y+3xy^2+y^3)\times(x^2+2xy+y^2)\times(x+y)$.

5. $(2x^3-3x^2y+2y^3)\times(2x^3+3xy^2+2y^3)$.

6. $(2x-5)^2$. 7. $(y+2y^2+3y^3)^2$. 8. $(2-3z-3z^2+2z^3)^2$.

9. $(x^3-3x^2y^2+3xy^4-y^6)\times(x^4-4x^3y^2+6x^2y^4-4xy^6+y^8)$.

10. $(x^3-2x^2+1)\times(2x^2-3x+4)\times(x+1)$.

11. $(x^2-mx+m^2)\times(x^2+mx+m^2)\times(x^4+m^2x^2+m^4)$.

12. $(ay^6+by^4z^3-cy^2z^6)\times(ay^5z^2-by^3z^5+cyz^8)$.

Show that

13. $x\times(x+1)\times(x+2)\times(x+3)+1=(x^2+3x+1)^2$.

14. $(y-1)\times y\times(y+1)\times(y+2)+1=(y^2+y-1)^2$.

Show that if y be replaced by $x+1$, the equation in ex. 14 becomes the equation in ex. 13.

15. By successive multiplications, and preferably by detached coefficients, find the first five powers of $a+b$ and of $a-b$.

16. So, of $x+y$ and of $x-y$, of $h+k$ and of $h-k$.

17. From observation, and comparison of the results above, state a general principle that holds good as to:

1. The number of terms in any power of a binomial.
2. The exponents of the first letter (a, x, or h) in the successive terms of any power.
3. The exponents of the last letter (b, y, or k).
4. The signs of the terms.
5. The coefficients of the first term, the second, and last.

18. Show that the terms of any power of $a+b$ are homogeneous, and that terms equidistant from the ends have the same, or opposite, coefficients.

What terms of powers of $a+b$ are identical with terms of like powers of $a-b$, and what are opposites?

SYMMETRY.

NOTE 6. The work is often shortened by noting any symmetry that may exist among the factors, singly or in groups.

E.g., the product $(2a+b+c)\times(a+2b+c)\times(a+b+2c)$, has the terms $2a\cdot a\cdot a=2a^3$,

$2a\cdot a\cdot b+2a\cdot 2b\cdot a\;+\;b\cdot a\cdot a=7a^2b$,

$2a\cdot 2b\cdot 2c+b\cdot c\cdot a+c\cdot a\cdot b+2a\cdot c\cdot b+b\cdot a\cdot 2c+c\cdot 2b\cdot a$
$=16abc$;

and $\because$ every term of the product, being entire and of the third degree, is of like form to one of these,

and the product is symmetric as to a, b, c;

$\therefore$ it has the terms $2b^3$, $2c^3$ as well as $2a^3$,

and $7b^2a$, $7c^2a$, $7ab^2$, $7bc^2$, $7ca^2$, as well as $7a^2b$,

and the whole product is

$$2a^3+2b^3+2c^3+7a^2b+7b^2c+7c^2a+7ab^2+7bc^2+7ca^2+16abc.$$

So, in the product $(2a+b-c)\times(2b+c-a)\times(2c+a-b)$:

the terms in a^3, b^3, c^3 have the same coefficient, -2,
those in a^2b, b^2c, c^2a have the same coefficient, 5,
those in ab^2, bc^2, ca^2 have the same coefficient, -1,
that in abc has the coefficient 2;

and the product is

$$-2(a^3+b^3+c^3)+5(a^2b+b^2c+c^2a)-(ab^2+bc^2+ca^2)+2abc.$$

So, to find the sum $(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2$:

by multiplication, or from the type-form $(a+b+\cdots)^2$, get

$$(a+b-2c)^2=a^2+b^2+4c^2+2ab-4bc-4ca,$$

by symmetry, write,

$$(b+c-2a)^2=b^2+c^2+4a^2+2bc-4ca-4ab,$$
$$(c+a-2b)^2=c^2+a^2+4b^2+2ca-4ab-4bc,$$

and add; the sum is $6(a^2+b^2+c^2-bc-ca-ab)$.

In such symmetric expressions, where three or more letters are involved, these letters may be kept advancing in cyclic order, abc, bca, cab, ab, bc, ca, as if they were points on a circle following one another in the same rotary direction.

QUESTIONS.

Multiply, add, and check the work:

1. $(x+y)^2+(x-y)^2$.
2. $(x+y)^2-(x-y)^2$.
3. $(x+y+z)^2+(x-y+z)^2$.
4. $(x+y+z)^2-(x-y+z)^2$.
5. $(x+y-z)^2+(x-y+z)^2$.
6. $(x+y-z)^2-(x-y+z)^2$.
7. $(-x+y+z)\times(x-y+z)\times(x+y-z)$.
8. $(-a+b+c+d)\cdot(a-b+c+d)\cdot(a+b-c+d)\cdot(a+b+c-d)$.
9. $(ax+by)\cdot(bx+ay)+(ax-by)\cdot(bx-ay)$.
10. $(a+b)\cdot(c+d)+(a+b)\cdot(c-d)+(a-b)\cdot(c+d)$
    $+(a-b)\cdot(c-d)$.
11. $(a+b+c)\cdot(x+y+z)+(a+b-c)\cdot(x+y-z)$
    $+(a-b+c)\cdot(x-y+z)+(-a+b+c)\cdot(-x+y+z)$.
12. $(bc+ad)^2+(ca+bd)^2+(ab+cd)^2$.

If $2s=a+b+c$, prove the identities below, and check the work by putting $a=b=c=2$.

13. $a^2-(b-c)^2=4(s-b)(s-c)$.
14. $(b+c)^2-a^2=4s(s-a)$.
15. $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$.
16. $(s-a)^3+(s-b)^3+(s-c)^3-s^3=-3abc$.
17. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$
    $+c(s-a)(s-b)=abc$.
18. $16s(s-a)(s-b)(s-c)=2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)$.
19. If $x+y+z=0$, show that $x^3+y^3+z^3=3xyz$; hence that
    $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)\cdot(c-a)\cdot(a-b)$.
20. $(x_1+x_2)\cdot(y_2-y_1)+(x_2+x_3)\cdot(y_3-y_2)+(x_3+x_1)\cdot(y_1-y_3)$
    $=x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3$.
21. $(ax+by+cz)\cdot(bx+cy+az)\cdot(cx+ay+bz)$
    $=abc(x^3+y^3+z^3)+(a^3+b^3+c^3)xyz+3abc\cdot xyz$
    $+(ab^2+bc^2+ca^2)\cdot(xy^2+yz^2+zx^2)$
    $+(a^2b+b^2c+c^2a)\cdot(x^2y+y^2z+z^2x)$.
22. $(ax+by+cz)^2+(ax+cy+bz)^2+(bx+ay+cz)^2$
    $+(bx+cy+az)^2+(cx+ay+bz)^2+(cx+by+az)^2$
    $=2(a^2+b^2+c^2)(x^2+y^2+z^2)+4(bc+ca+ab)(yz+zx+xy)$.

CONTRACTION.

Note 7. When only the first few terms of a product are wanted, the work is shortened by omitting all partial products that do not enter into the required terms.

E.g., to find $(1-3x+5x^2-\cdots)^2$ as far as the x^2-term:

$$
\begin{array}{r}
1-3x+5x^2-\cdots \\
1-3x+5x^2-\cdots \\
\hline
1-3x+5x^2 \\
-3x+9x^2 \\
+5x^2 \\
\hline
1-6x+19x^2
\end{array}
\qquad \text{or} \qquad
\begin{array}{r}
1-3+5-\cdots \\
1-3+5-\cdots \\
\hline
1-3+5 \\
-3+9 \\
+5 \\
\hline
1-6+19
\end{array}
$$

So, omitting x^4 and higher powers, to find the product,

$$(1+x+x^2+\cdots)\times(1-2x+3x^2-\cdots)\times(1+4x+9x^2+\cdots):$$

write

$$
\begin{array}{rrrr}
\overline{1} & \overline{1} & \overline{1} & \overline{1} \\
 & {}^-2 & {}^-2 & {}^-2 \\
 & & 3 & 3 \\
 & & & {}^-4 \\
\hline
1 & {}^-1 & 2 & {}^-2
\end{array}
\times \overline{1\ \ {}^-2\ \ 3\ \ {}^-4}
\qquad
\begin{array}{rrrr}
\overline{1} & \overline{{}^-1} & \overline{2} & \overline{{}^-2} \\
 & 4 & {}^-4 & 8 \\
 & & 9 & {}^-9 \\
 & & & 16 \\
\hline
1 & 3 & 7 & 13
\end{array}
\times \overline{1\ \ 4\ \ 9\ \ 16}
$$

and the product sought is $1+3x+7x^2+13x^3$.

This method of contracted multiplication may be used, with great profit, with decimal fractions.

E.g., to find the product 37.8562×14.9716, correct to two places, and $.2819\times .3781\times .2148$ to three places.

$$
\begin{array}{r}
37.8562 \\
14.9716 \\
\hline
378.562 \\
151.425 \\
34.070 \\
2.650 \\
38 \\
23 \\
\hline
566.77
\end{array}
\qquad \text{and} \qquad
\begin{array}{r}
.2819 \\
.3781 \\
\hline
.0846 \\
197 \\
22 \\
\hline
.1065
\end{array}
\qquad\qquad
\begin{array}{r}
.1065 \\
.2148 \\
\hline
.0213 \\
11 \\
4 \\
1 \\
\hline
.023
\end{array}
$$

In writing down the partial products, carry what would have been carried had the multiplication been made in full. E.g, the partial product $23=3\cdot 6+5$ carried from $78\cdot 6$.

QUESTIONS.

Find the products (or powers) as far as the x^4-term:

1. $(1+x+x^2+\cdots)\cdot(1-x+x^2-\cdots)$.
2. $(1+2x+3x^2+\cdots)\cdot(1-2x+3x^2-\cdots)$.
3. $(1+3x+5x^2)\cdot(1+3x)\cdot(1+5x^2)$.
4. $(3-5x+9x^2)\cdot(1-5x)\cdot(1+9x^2)$.
5. $(1-2x+3x^2-4x^3+5x^4)^2$. 6. $(1-\frac{1}{2}x+\frac{2}{3}x^2-\frac{3}{4}x^3+\frac{4}{5}x^4)^2$.
7. $(3+5x+7x^2+\cdots)^3$. 8. $(\frac{1}{3}-\frac{1}{5}x+\frac{1}{7}x^2-\cdots)^3$.
9. $(a+bx+cx^2+\cdots)\cdot(a'+b'x+c'x^2+\cdots)$.
10. $(a-bx+cx^2-\cdots)\cdot(a'-b'x+c'x^2-\cdots)$.
11. $(a+bx+cx^2+dx^3+\cdots)^2$. 12. $(a-bx+cx^2-dx^3+\cdots)^2$.
13. $(a+bx+cx^2+dx^3+\cdots)^3$. 14. $(a-bx+cx^2-dx^3+\cdots)^3$.

Find the products (or powers) as far as the x^3-term, with three-figure decimals:

15. $(1+.5x+.09x^2)\cdot(1-.5x+.09x^2)$.
16. $(3+.5x-.07x^2)\cdot(3-.5x+.07x^2)$.
17. $(1-.07x^2)\cdot(3+.009x^3)$. 18. $(1+.007x^3)\cdot(3-.09x^2)$.
19. $(1+.167x+.014x^2+.001x^3)^2$.
20. $(1-.167x+.014x^2-.001x^3)^2$.
21. $(1+.056x^2-.006x^3)^3$. 22. $(1-.333x+.006x^2)^3$.
23. $(3+.5x+.07x^2+.009x^3)\cdot(5x+.07x^2)\cdot(7+.009x^2)$.
24. $(1+.07x)\cdot(1+.07x^2)\cdot(1+.07x^3)$.
25. $(1+.07x^2)^2$. 26. $(1+.07x^3)^3$. 27. $(1+.07x^4)^4$.

Find the values correct to thousandths, when $x=.1$:

28. $(1+2x)^3$. 29. $(1+2x+3x^2)^3$. 30. $(1+2x+3x^2+4x^3)^3$.
31. $(x+5)^3$. 32. $(x^2-x+5)^3$. 33. $(x^3-x-5)^3$.

Find the values correct to thousandths when $x=.02$:

34. $(1+2x)^3$. 35. $(1+2x+3x^2)^3$. 36. $(1+2x+3x^2+4x^3)^3$.
37. $(x+5)^4$. 38. $(x^2-x+5)^4$. 39. $(x^3-x-5)^4$.
40. If $x=a+by+cy^2+dy^3+\cdots$, $y=l+mz+nz^2+pz^3+\cdots$, find the value of x in terms of z as far as the z^3-term.

§ 4. DIVISION.

PROB. 4. TO DIVIDE ONE NUMBER BY ANOTHER.

(*a*) *A monomial by a monomial: to the quotient of the numerical coefficients, annex the several literal factors, each taken as many times as the excess of the exponent of the dividend over that of the divisor;* [I th. 10 cr.

mark the quotient positive if the elements be both positive or both negative, and negative if one element be positive and the other negative. [inv. pr. 3.

E.g., $63a^{-2}b^2d^5 : 7ac^3d^5 = 9a^{-3}b^2c^{-3}$,

$-35x^4y^{-4}z^4 : 5xy^{-2}z^5 = -7x^3y^{-2}z^{-1}$,

$\frac{63}{25}a^{-2}b^{-2}d^{-5} : -\frac{7}{5}ac^{-3}d^{-5} = -\frac{9}{5}a^{-3}b^{-2}c^3$,

$-\frac{2}{3}x^{-4}y^3z^{-3} : -\frac{7}{2}x^{-5}y^{-2}z^3 = \frac{4}{21}xy^5z^{-6}$.

(*b*) *A polynomial by a monomial: divide each term of the dividend by the divisor; add the partial quotients.* [I th. 7.

E.g., $(45x^3y^2z + 105xz^{-1} - 165x^{-2}y^{-3}z^{-4}) : -15xy^2z^{-2}$

$= -3x^2z^3 - 7y^{-2}z + 11x^{-3}y^{-5}z^{-2}$.

(*c*) *A polynomial by a polynomial: arrange the terms of both polynomials as to the powers of some one letter;*

divide the first term of the dividend by the first term of the divisor; multiply the whole divisor by this partial quotient, and subtract the product from the dividend;

repeat the work upon the remainder as a new dividend;

add the partial quotients; their sum is the quotient sought, and the part of the dividend left undivided is the *remainder.* [I th. 7 cr.

E.g., to divide a^3+b^3 by $a+b$:

write
$$\begin{array}{l|l}
a^3+b^3 & a+b \\
\underline{a^3+a^2b} & \overline{a^2-ab+b^2} \\
\quad -a^2b+b^3 & \\
\quad \underline{-a^2b-ab^2} & \\
\qquad\quad ab^2+b^3 & \\
\qquad\quad \underline{ab^2+b^3} &
\end{array}$$

QUESTIONS.

Find the quotients below, and check the work:

1. $3a^2b : ab$. 2. ${}^-3ax : {}^-x^2$. 3. $mn^{-2} : {}^-m^2n$.

4. ${}^-r^3st^{-1} : 2r^{-2}s^2t^{-2}$. 5. $51a^3b^2c : {}^-17a^2b$. 6. $231x^{n+2}y^4 : 3x^ny$.

7. $(x^2+2ax+b) : x$. 8. $(\frac{1}{2}x^2-\frac{2}{3}xy^{-2}+\frac{3}{4}y^{-4}) : -3x^3y^{-2}$.

9. $(x^2-x-12) : (x-4)$. 10. $(4x^4-12x^2+9) : (2x^2-3)$.

11. $(15x^3+x^2y^{-1}+4y^{-3}) : (3x+2y^{-1})$.

12. $(a^2x^4-y^6) : (ax^2-y^3)$. 13. $(x^4-2x^2y-3y^2) : (x^2+y)$.

14. $(x^{2n}-a^{2n}) : (x^n+a^n)$. 15. $(9a^4x^6-49c^2y^4) : (3a^2x^3-7cy^2)$.

16. $(1-x^4) : (1+x)$. 17. $(c^6-3c^4+3c^2-1) : (c^2-1)$.

18. $(a^{m+n}-a^mb^n+a^nb^m-b^{m+n}) : (a^n-b^n)$.

19. $(ar^n-a) : (r-1)$; $(a-ar^n) : (1-r)$.

20. $(a^2-2ab-2ac+b^2+2bc+c^2) : (a-b-c)$.

21. $(a^2-b^2+2bc-c^2) : (a+b-c)$; $(b^2+2bc+c^2-a^2) : (b+c-a)$.

22. $(a^2-m^2x^2+2mnx^3-n^2x^4) : (a-mx+nx^2)$.

23. $(a^5-x^5) : (a-x)$; $(a^6-y^6) : (a-y)$; $(x^6-y^6) : (x^2-y^2)$.

24. $(a^5+x^5) : (a+x)$; $(a^6-y^6) : (a+y)$; $(x^6+y^6) : (x^2+y^2)$.

25. $(x^{n+2}+3x^3-4x^{n-2}+6x) : (x^2+2)$.

26. $(x^2+ax+bx+cx+dx+ac+ad+bc+bd) : (x+a+b)$.

27. $(12m^3x^3-17m^2x^2+10mx-3) : (4mx-3)$.

28. $(a^{n+2}-3a^{n+1}-a^n+2a^{n-1}-4a^{n-2}) : (a^2-a+1)$.

29. $(2y^2+3y-7+15y^{-1}-8y^{-2}+5y^{-3}) : (2-y^{-1}+y^{-2})$.

30. $(x^6-22x^4+60x^3-55x^2+12x+4) : (x^2+6x+1)$.

31. $(x^2y+xy^2+x^2z+2xyz+y^2z+xz^2+yz^2) : (y+z) : (x+y)$.

32. $(x^4-4x^3y+6x^2y^2-4xy^3+y^4)\cdot(x^2+2xy+y^2)$
$: (x^4-2x^3y+2xy^3-y^4)$.

33. $$\overline{\begin{array}{l|l|l|l} x^4+a & x^3+ab & x^2+abc & x+abcd \\ \quad\;\, +b & \quad\;\, +ac & \quad\;\, +abd & \\ \quad\;\, +c & \quad\;\, +ad & \quad\;\, +acd & \\ \quad\;\, +d & \quad\;\, +bc & \quad\;\, +bcd & \\ & \quad\;\, +bd & & \\ & \quad\;\, +cd & & \end{array}} : \overline{\begin{array}{l|l} x^2+a & x+ab \\ \quad\;\, +b & \end{array}}$$

CHECKS.

The work is tested by multiplication and sometimes by the principles of prob. 3, note 1.

FORM OF QUOTIENT.

Note 1. With some changes that are apparent, the principles of prob. 3, note 1 hold true in division.

ARRANGEMENT.

Note 2. Unless the division be exact, the quotient and remainder are different for every different choice of trial divisor; but the complete quotient, including the fraction, is always the same.

E.g., $(x^2+1):(x+1)$ gives for quotient and remainder:

$x-1,\quad 2\quad$ when x is trial divisor,

$1-x,\quad 2x^2\quad$ when 1 is trial divisor.

So, $(x^2+y^2+z^2):(x+y+z)$ gives for quotient and remainder:

$x-y-z,\quad 2(y^2+yz+z^2)\quad$ when x is trial divisor,

$y-z-x,\quad 2(z^2+zx+x^2)\quad$ when y is trial divisor,

$z-x-y,\quad 2(x^2+xy+y^2)\quad$ when z is trial divisor.

DETACHED COEFFICIENTS.

Note 3. If both dividend and divisor be such that, when their coefficients are detached, the remaining factors of pairs of successive terms have a constant ratio, the work is shortened by writing down these coefficients.

In place of the given polynomials, write the two groups of coefficients, with 0 for the coefficient of any absent term;

divide these coefficients as in the division of polynomials, and restore the suppressed letters.

E.g., $(a^3+b^3):(a+b)$ gives

$$
\begin{array}{rrrr|l}
1 & 0 & 0 & 1 & 1\ \ 1 \\ \cline{5-5}
1 & 1 & & & 1\ {}^{-}1\ \ 1 \\ \cline{1-4}
 & {}^{-}1 & 0 & 1 & \\
 & {}^{-}1 & {}^{-}1 & & \\ \cline{2-4}
 & & 1 & 1 &
\end{array}
$$

and the quotient is a^2-ab+b^2.

QUESTIONS.

1. Show that the rules for the coefficients, exponents, and signs of a quotient are a direct consequence of division's being the inverse of multiplication.

2. State the first six of the principles in prob. 3, note 1, in such form that they will apply to division.

3. Show, as a consequence of the seventh principle, that a monomial is not divisible by a polynomial, and that when there is no remainder, the first and last terms of the dividend are divisible by the first and last terms of the divisor.

4. Find the quotient $(1+x^3-8y^3+6xy):(1+x-2y)$, having it arranged in the order of rising powers of x, and the coefficients to falling powers of y.

5. So, $(18xyz+27z^3-x^3+8y^3):(x-3z-2y)$.

6. By detached coefficients, divide $x^4+10x^3+35x^2+50x+24$ in turn by $x+1$, $x+2$, $x+3$, $x+4$, and check the work.

7. So, x^4-5x^2+4 by $x-2$, $x-1$, $x+1$, $x+2$.

8. So, $x^4-10x^3+35x^2-50x+24$ by $x-1$, $x-2$, $x-3$, $x-4$.

9. So, $x^4-5x^2y^2+4y^4$ by $x-2y$, $x-y$, $x+y$, $x+2y$.

10. So, $x^4-10x^3y+35x^2y^2-50xy^3+24y^4$ by $x-y$, $x-2y$, $x-3y$, $x-4y$.

11. So, $x^4+4x^3-6x^2+7x-2$ by x^2+5x-2, x^2-x+1.

12. So, $a^4+4a^3b-5a^2b^2+6ab^3-b^4$ by $a^2+5ab-b^2$, a^2-ab+b^2.

13. Divide $x^5-2x^4+3x^3-4x^2+5x-6$ by $x-2$, the quotient by $x-2$, and so on. Write the last quotient and the numerical remainders in their order, as coefficients of powers of $x-2$.

14. So, $x^5+2x^4+3x^3+4x^2+5x+6$ by $x+2$, the quotient by $x+2$, and so on.

15. So, $x^5+2x^4y+3x^3y^2+4x^2y^3+5xy^4+6y^5$ by $x+2y$, the quotient by $x+2y$, and so on.

16. Compare the work in exs. 13–15 with that of reducing seconds to minutes, hours, and days by the ordinary process of reduction ascending: what is the scale in ex. 14? in ex. 15?

TYPE-FORMS.

Note 4. If the dividend and divisor can be reduced to known type-forms [pr. 3, nt. 4] the quotient may be written directly.

E.g., $(x^2+7x+12):(x+4)=x+3$, [1

$(x^2-y^2):(x+y)=x-y$, [2

$(x^8-y^8):(x^2-y^2)=x^6+x^4y^2+x^2y^4+y^6$. [6

SYMMETRY.

Note 5. If both the dividend and divisor be symmetric as to two or more letters, and if there be no remainder, then the quotient will be a symmetric function of the same letters. It is then often sufficient to get a few characteristic terms, and to write the rest therefrom by symmetry.

E.g., $(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2+3xyz):(x+y+z)$,
gives xy for one term of the quotient;

$\therefore$ yz and zx are also necessary terms of the quotient;

and $\because$ the product $(x+y+z)\cdot(xy+yz+zx)$ is the dividend,

$\therefore$ $xy+yz+zx$ is the quotient sought.

The pupil must, however, use great caution if he employ symmetry in division; he may use it as suggestive of the true answer, but hardly ever as conclusive, because he cannot be sure beforehand that there is no remainder.

CONTRACTION.

Note 6. When only the first few terms of a quotient are wanted, the work is shortened by omitting all partial products that do not affect the required terms.

E.g., $(1+x+x^2+x^3):(1-2x+3x^2-4x^3)$ to four terms gives

$$\begin{array}{r|l}
1+\ x+\ x^2+\ x^3 & 1-2x+3x^2-4x^3 \\
1-2x+3x^2-4x^3 & \overline{1+3x+4x^2+4x^3} \\
\hline
3x-2x^2+5x^3 & \\
3x-6x^2+9x^3 & \\
\hline
4x^2-4x^3 & \\
4x^2-8x^3 & \\
\hline
4x^3 &
\end{array}
\qquad
\begin{array}{rrrr|rrrr}
1 & 1 & 1 & 1 & 1 & -2 & 3 & -4 \\
1 & -2 & 3 & -4 & 1 & 3 & 4 & 4 \\
\hline
 & 3 & -2 & 5 & & & & \\
 & 3 & -6 & 9 & & & & \\
\hline
 & & 4 & -4 & & & & \\
 & & 4 & -8 & & & & \\
\hline
 & & & 4 & & & &
\end{array}$$

QUESTIONS.

Name the type-forms, find the quotients, and check the work:

1. $(x^2-21x+104):(x-13)$.
2. $(x^4-y^4):(x^2-y^2)$.
3. $(x^2-4x-12):(x-6)$.
4. $(130+31xy+x^2y^2):(5+xy)$.
5. $(2+x-x^2):(2-x)$.
6. $(2x^2-x-15):(x-3)$.
7. $(4x^6-9y^6):(2x^3+3y^3)$.
8. $(a^2b^2-49x^4):(ab-7x^2)$.
9. $(x^{2n}-y^{2n}):(x^n \pm y^n)$.
10. $(1-100a^4b^2c^6):(1+10a^2bc^3)$.
11. $[1-(7a-3b)^2]:(1+7a-3b)$.
12. $[(x^3+a^3)\cdot(x^3-a^3)]:[(x^2+ax+a^2)\cdot(x^2-ax+a^2)]$.
13. $[(a+b)^2-c^2]:(a+b-c)$.
14. $[a^2-(b-c)^2]:(a-b+c)$.
15. $(a^2+2ax^2+x^4):(a+x^2)$.
16. $(x^2-8x+16):(x-4)$.
17. $(4x^2-12xy+9y^2):(2x-3y)$.
18. $(49a^2+42ab+9b^2):(7a+3b)$.
19. $(4a^2+b^2+c^2-4ab+4ac-2bc):(2a-b+c)$.
20. $(a^3b^3-343x^3):(ab-7x)$.
21. $(a^3+216b^3):(a+6b)$.
22. $[(x+y)^3+z^3]:(x+y+z)$.
23. $[x^3-(y-z)^3]:(x-y+z)$.
24. $(a^{2n}-b^{2n}):(a+b)$.
25. $(a^{2n+1}+b^{2n+1}):(a+b)$.
26. $(x^{mn}-1):(x^m-1)$.
27. $(x^{mn}-1):(x^n-1)$.

By symmetry, find the quotients below, and check the work:

28. $[3abc+a^2(b+c)+b^2(c+a)+c^2(a+b)]:(a+b+c)$.
29. $(a^3+b^3+c^3-3abc):(a+b+c)$.
30. $(a^3-b^3+c^3+3abc):(a-b+c)$.
31. $(a^8+a^6b^2+a^4b^4+a^2b^6+b^8):(a^4+a^3b+a^2b^2+ab^3+b^4)$.
32. $(a^4b^2-a^2b^4+b^4c^2-b^2c^4+c^4a^2-c^2a^4)$
 $:(a^2b-ab^2+b^2c-bc^2+c^2a-ca^2)$.
33. $[x^3(y-z)+y^3(z-x)+z^3(x-y)]$
 $:[x^2(y-z)+y^2(z-x)+z^2(x-y)]$.

Find the quotients to the x^4-term, using three-place decimals:

34. $(1-.2x+.04x^2-\cdots):(1+.1x+.01x^2+\cdots)$.
35. $(1+.2x+.04x^2+\cdots):(1-.1x+.01x^2-\cdots)$.
36. $(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots):(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots)$.

SYNTHETIC DIVISION.

Note 7. If the coefficient of the first term of the divisor be 1, the work of division by detached coefficients is shortened by the method of *synthetic division.* It is a species of short division, in which the same numbers play the part of both remainders and quotients.

E.g., to divide x^3+5x^2+8x+4 by $x+2$:

$$\begin{array}{r|rrrr} \text{write} & 1 & 5 & 8 & 4 \\ 2 & & 2 & 6 & 4 \\ \cline{2-5} 1 & 1 & 3 & 2, & 0 \end{array}$$

The divisor stands at the left in a vertical column, reading upward; the part products 2, 6, 4 are directly below the like terms of the dividend; of the figures 1, 3, 2, 0, in the lower line, 1 is the first quotient figure, 3 is first a remainder standing for $3x^2$, then a quotient figure standing for $3x$, 2 is first a remainder standing for $2x$, then a quotient figure, and 0 is the final remainder. The quotient is x^2+3x+2.

It is customary to change the signs of the terms of the divisor after the first, and so of the resulting part products; the subtractions are then done by addition, and the work takes the form

$$\begin{array}{r|rrrr} & 1 & 5 & 8 & 4 \\ {}^-2 & & {}^-2 & {}^-6 & {}^-4 \\ \cline{2-5} 1 & 1 & 3 & 2, & 0 \end{array}$$

So, $(x^3+5x^2+9x+6):(x^2+3x+2)$ gives

$$\begin{array}{r|rrrr} & 1 & 5 & 9 & 6 \\ {}^-2 & & & {}^-2 & {}^-4 \\ {}^-3 & & {}^-3 & {}^-6 & \\ \cline{2-5} 1 & 1 & 2, & 1 & 2 \end{array}$$

wherein both 3 and 2 have their signs changed, and their products by 1, the first quotient figure, give the ${}^-3$, ${}^-2$ in the body of the work; the sum $5+{}^-3$ is 2, the first remainder and the second quotient figure; and ${}^-6$, ${}^-4$ are the part products of ${}^-3$, ${}^-2$ by this 2.

The quotient is $x+2$, and the final remainder is $x+2$.

QUESTIONS.

1. Show that, with a divisor in which the coefficient of the first term is unity, each term of the quotient has the same coefficient as the first term of the dividend used: *i.e.*, that the same number plays a double part, being the coefficient of the first term of the remainder and of the new term in the quotient, and that it need, therefore, be written but once.

2. Show that changing the sign of the divisor and adding the partial products to the dividend does not change the value of the remainder, and therefore not that of the quotient.

By synthetic division, find the quotients, and check the work:

3. $(x^3-5x^2-46x-40):(x+4)$. 4. $(x^3-40):(x+4)$.

5. $(x^4+x^3-5x^2-11x-30):(x-3)$. 6. $(x^4-5x^2-30):(x-3)$.

7. $(3x^3-5x^2-7x+10):(x-2)$. 8. $(3x^2-7x+10):(x-2)$.

9. $(4x^5+17x^4+9x^3-20x^2-3x+9):(x+3)$.

10. $(x^7-1):(x-1)$. 11. $(x^6-y^6):(x-y)$.

12. $(x^6-y^6):(x+y)$. 13. $(x^3-x^2y+xy^2-y^3):(x-y)$.

14. $(x^3-3x^2y+3xy^2-y^3):(x-y)$. 15. $(6x^4-96):(x-2)$.

16. $(a^4-a^2b^2+2ab^3-b^4):(a^2-ab+b^2)$. 17. $(a^4-b^4):(a^2+b^2)$.

18. $(x^6+10x-33):(x^2-2x+3)$. 19. $(x^6+10x-33):(x^2+3)$.

20. $(3x^5-4x^4-x^3+23x^2-28x+15):(x^2-2x+3)$.

21. $(a^4+a^3b-8a^2b^2+19ab^3-15b^4):(a^2+3ab-5b^2)$.

22. $(4z^5-18z^3-16z^2-78z+54):(z^3-2z^2+z-9)$.

23. $(4x^6-5x^5+8x^4-10x^3-8x^2-5x-4):(x^3-2x^2+3x-4)$.

24. $(x^5+151x-264):(x^2-4x+11)$. 25. $(x^5-264):(x^2+11)$.

26. $(2x^5-82x-240):(x^2+4x+5)$. 27. $(2x^5-240):(x^2+5)$.

28. $(x^4+x^3-4x^2+5x-3):(x^2+2x-3)$. 29. $(x^4-3):(x^2-3)$.

30. $(x^6-22x^4+60x^3-55x^2+12x+4):(x^2-3x+2)$.

31. Divide $3x^4-5x^3+7x^2-9x+11$ by $x-3$, the quotient by $x-3$, and so on; write the last quotient and the remainders as coefficients of powers of $(x-3)$.

32. So divide $3x^4+5x^3+7x^2+9x+11$ by $x+3$, by $x+3,\cdots$

§ 5. FRACTIONS.

PROB. 5. TO REDUCE A SIMPLE FRACTION TO LOWER TERMS.

Divide both terms by any entire number that divides them without remainder; the quotients are the terms of the reduced fraction. [I th. 2 cr. 4.

E.g., $36a^4b^2c^3/24a^5bx = 3bc^3/2ax$. [div. $12\,a^4b$.

For reduction of fractions to their lowest terms, see IV, pr. 4.

PROB. 6. TO REDUCE A SIMPLE FRACTION TO AN EQUAL FRACTION HAVING A GIVEN NUMERATOR OR DENOMINATOR.

Divide the given new numerator or denominator by the old one, and multiply both terms of the fraction by the quotient. [I th. 2 cr. 4.

E.g., to reduce $3x^2y/2a^2b$ to an equal fraction with denominator $2a^3bc$,

then $2a^3bc : 2a^2b = ac$, and the fraction sought is $3acx^2y/2a^3bc$.

So, to reduce $2x^2z/3a^2c$ to an equal fraction with numerator $6x^2yz$,

then $6x^2yz : 2x^2z = 3y$, and the fraction sought is $6x^2yz/9a^2cy$.

So, entire and mixed numbers are reduced to simple fractions.

E.g., $x + 2a = (dx + 2ad)/d$, $\quad x - 2a + a^2/d = (dx - 2ad + a^2)/d$.

PROB. 7. TO REDUCE TWO OR MORE SIMPLE FRACTIONS TO EQUAL FRACTIONS HAVING A COMMON DENOMINATOR.

Over the continued product of the denominators write the product of each numerator into all the denominators except its own, or [I th. 2 cr. 4.

find some number that can be exactly divided by all the denominators;

divide this number by the denominators in turn and multiply each numerator by the quotient got by using its denominator as divisor.

E.g., $\dfrac{5xy}{2a}, \ \dfrac{3bc}{x}, \ \dfrac{3(a-b)}{7} = \dfrac{35x^2y}{14ax}, \ \dfrac{42abc}{14ax}, \ \dfrac{6ax(a-b)}{14ax}$.

For finding the lowest common denominator, see IV, pr. 5.

QUESTIONS.

Reduce the fractions below to lower terms:

1. $\dfrac{x^2-3x+2}{x^2-4x+3}$. 2. $\dfrac{x^2-2x-15}{x^2+2x-35}$. 3. $\dfrac{acx^2+(ad-bc)x-bd}{a^2x^2-b^2}$.

4. $\dfrac{a^2-b^2}{a^4-b^4}$. 5. $\dfrac{a^2-b^2}{a^2\pm 2ab+b^2}$. 6. $\dfrac{4x^2-9}{4x^2\pm 12x+9}$.

7. $\dfrac{4x^2-(3y-4z)^2}{(2x+3y)^2-16z^2}$. 8. $\dfrac{(4x^2+3x+2)^2-(2x^2+3x+4)^2}{(3x^2+x-1)^2-(x^2-x-3)^2}$.

9. $\dfrac{m^3-n^3}{m^4-n^4}$. 10. $\dfrac{p^4-q^4}{p^5-q^5}$. 11. $\dfrac{r^5-s^5}{r^6-s^6}$. 12. $\dfrac{x^{2n}-y^{2n}}{x^{3n}-y^{3n}}$.

Change the fractions below to equal fractions:

13. $x/(x-3)$ with denominator x^2-5x+6.

14. $(4a-3)/(4a-4)$ with denominator $16a^2-28a+12$.

15. $(a-b)/(a^{n-1}+a^{n-2}b+\cdots+b^{n-1})$ with denominator a^n-b^n.

16. $(x-5)/(x^2-1)$ with denominator $1-x^4$.

17. $(b-a)/(2x+3)(3-2x)$ with denominator $4x^2-9$.

18. $a/(a-c)(b-c)$, $b/(a-c)(c-b)$, $c/[c^2-(a+b)c+ab]$ with denominator $(b-c)(c-a)$.

19. Reduce the fractions below to equal fractions, with the common numerator a^4-b^4:

$$\frac{a^2-b^2}{a^2+b^2},\quad \frac{a^2+b^2}{a^2-b^2},\quad \frac{a^3+a^2b+ab^2+b^3}{a^3-a^2b+ab^2-b^3},\quad \frac{a^3-a^2b+ab^2-b^3}{a^3+a^2b+ab^2+b^3}.$$

Reduce to equal fractions having common denominators:

20. $\dfrac{x}{1-x^2}$, $\dfrac{x}{(1-x)^2}$, $\dfrac{x}{(1+x)^2}$. 21. $\dfrac{x^2}{a^2+ax}$, $\dfrac{a^2}{x^2-ax}$, $\dfrac{ax}{a^2-x^2}$.

22. $\dfrac{a}{a-x}$, $\dfrac{3a}{a+x}$, $\dfrac{2ax}{a^2-x^2}$. 23. $\dfrac{2}{x}$, $\dfrac{3}{2x-1}$, $\dfrac{2x-3}{4x^2-1}$.

24. $\dfrac{a^2-bc}{(a-b)(a-c)}$, $\dfrac{b^2-ca}{(b-c)(b-a)}$, $\dfrac{c^2-ab}{(c-a)(c-b)}$.

25. $\dfrac{2}{(x-1)(x-2)}$, $\dfrac{5}{x^2-5x+6}$, $\dfrac{3}{x^2-4x+3}$.

26. $\dfrac{r^2-r-2}{r^2+r+1}$, $\dfrac{r^2+r-1}{1-r+r^2}$, $\dfrac{r^3-1}{1-r}$, $\dfrac{r^3+1}{1+r}$, $\dfrac{1-r}{r^3-1}$, $\dfrac{r+1}{1+r^3}$.

PROB. 8. TO ADD FRACTIONS.

Reduce the several fractions to equal fractions having a common denominator; [pr. 7.
write the sum of the new numerators over the common denominator.

E.g., $$\frac{3bc^3}{2ax}+\frac{3(a-b)}{7}=\frac{21bc^3+6ax(a-b)}{14ax}.$$

Subtraction is but a case of addition; *add the opposite of the subtrahend.*

E.g., $$\frac{3bc^3}{2ax}-\frac{3(a-b)}{7}=\frac{21bc^3-6ax(a-b)}{14ax}.$$

PROB. 9. TO MULTIPLY FRACTIONS.

Write the product of the numerators over the product of the denominators, cancelling any factor that is common to a numerator and a denominator. [I th. 2 crs. 2, 4.

E.g., $$\frac{3bc^3}{2ax}\times\frac{3(a-b)}{7}=\frac{9bc^3(a-b)}{14ax};\quad \frac{3bc^3}{2ax}\times\frac{8ay}{9b^2c^2}=\frac{4y}{3bx}.$$

Division is but a case of multiplication; *multiply by the reciprocal of the divisor.*

E.g., $$\frac{3bc^3}{2ax}:\frac{3(a-b)}{7}=\frac{3bc^3}{2ax}\times\frac{7}{3(a-b)}=\frac{7bc^3}{2ax(a-b)}.$$

A *complex fraction* is an indicated division wherein the dividend, the divisor, or both, are simple fractions.

E.g., $$\frac{\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}=\frac{\dfrac{(x^2+y^2)^2-(x^2-y^2)^2}{(x^2-y^2)\cdot(x^2+y^2)}}{\dfrac{(x+y)^2-(x-y)^2}{(x-y)\cdot(x+y)}}=\frac{\dfrac{4x^2y^2}{x^4-y^4}}{\dfrac{4xy}{x^2-y^2}}$$

$$=\frac{4x^2y^2}{x^4-y^4}\times\frac{x^2-y^2}{4xy}=\frac{xy}{x^2+y^2}.$$

This example may also be worked by multiplying both terms of the complex fraction by x^4-y^4.

QUESTIONS.

Add, subtract, multiply, and divide as shown below:

1. $1+\frac{1}{1+x}+\frac{1}{1-x}$. 2. $1-\frac{1}{1+x}-\frac{1}{1-x}$. 3. $\frac{1-x}{1+x}+\frac{1+x}{1-x}$.

4. $\frac{a+b}{a+x}-\frac{a-b}{a-x}+\frac{a+b}{a-x}-\frac{a-b}{a+x}$. 5. $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$.

6. $\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

7. $\frac{a^3}{(a-b)\cdot(a-c)}+\frac{b^3}{(b-c)\cdot(b-a)}+\frac{c^3}{(c-a)\cdot(c-b)}$.

8. $\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{x^n}{x^n-1}+\frac{x^0}{x^n+1}$. 9. $1+\frac{x}{y+z}+\frac{3-a}{y-z}+\frac{5-a}{z-y}$.

10. $\frac{y^2c^2}{b^2c^2}+\frac{(y^2-b^2)\cdot(z^2-b^2)}{b^2(b^2-c^2)}+\frac{(y^2-c^2)\cdot(z^2-c^2)}{c^2(c^2-b^2)}$.

11. $\frac{a+1}{a(a-b)(a-c)}+\frac{b+1}{b(b-c)(b-a)}+\frac{c+1}{c(c-a)(c-b)}$.

12. $\frac{a^2-b^2}{x-y}\cdot\frac{x^2-y^2}{a-b}\cdot\frac{c^2}{x+y}$. 13. $\left(1+\frac{1}{x}\right):\left(x-\frac{1}{x}\right)\cdot\left(1-\frac{1}{x}\right)$.

14. $\frac{x^4-b^4}{x^2-2bx+b^2}:\frac{x^2+bx}{x-b}\cdot\frac{x^5-b^2x^3}{x^3+b^3}:\frac{x^4-2bx^3+b^2x^2}{x^2-bx+b^2}$.

15. $\frac{a^3-x^3}{a^3+x^3}\cdot\frac{a^2-x^2}{a^2+x^2}\cdot\frac{a-x}{a+x}\cdot\frac{a^2-ax+x^2}{a^2+ax+x^2}\cdot\frac{a^2+2ax+x^2}{a^2-2ax+x^2}$.

16. $\frac{x+1}{y}\cdot\frac{y^2+2y+1}{x^2-1}\cdot\frac{(x-1)y}{(y+1)^2}$. 17. $\frac{a+b}{m}:\frac{b^2-a^2}{m^2}\cdot\left(\frac{1}{x}:\frac{m}{a-b}\right)$.

18. $\frac{a}{b+c}\cdot\frac{b}{a+c}\cdot\frac{c}{a+b}:\frac{abc}{(a^3+b^3)(a^3+c^3)}:\frac{a^2-ab+b^2}{c^2-b^2}$.

19. $\left[\left(x+\frac{x+1}{x}\right):\left(1+\frac{a}{b}\right)\right]:\left[\frac{1}{2}b:\left(1+\frac{b}{a}\right):4ax:\left(x+\frac{1}{x+1}\right)\right]$.

20. Reduce the complex fractions below to simple fractions:

$$\frac{x-\frac{x-y}{1+xy}}{1+\frac{x(x-y)}{1+xy}};\quad \frac{\frac{m^2+mn+n^2}{m^3+n^3}}{\frac{m^3-n^3}{m^2-mn+n^2}};\quad \frac{\frac{p^6+q^6}{p^3+q^3}}{\frac{p^6-q^6}{p^3-q^3}};\quad \frac{\frac{m+n}{m-n}+\frac{m-n}{m+n}}{\frac{m-n}{m+n}-\frac{m+n}{m-n}}.$$

§ 6. QUESTIONS FOR REVIEW.

Define and illustrate:

1. An algebraic expression; a binomial; a trinomial; a quadrinomial; a polynomial.

2. Expressions that are literal; numerical; entire; fractional; symmetric; homogeneous.

3. A series; a finite series; an infinite series.

4. The degree of a term, and of a polynomial; a coefficient; like terms; unlike terms.

Give the general rule, with reasons and illustrations, for:

5. Adding like numbers; unlike numbers.

6. Subtracting one number from another.

7. Multiplying a monomial by a monomial; a polynomial by a monomial; a polynomial by a polynomial.

8. Dividing a monomial by a monomial; a polynomial by a monomial; a polynomial by a polynomial.

9. Reducing a simple fraction to lower terms; to an equal fraction having a given numerator or denominator.

10. Reducing two or more simple fractions to equal fractions having a common denominator.

11. Adding and subtracting fractions.

12. Multiplying and dividing fractions.

State the principles that relate to the form of a product:

13. As to its independence of the values of the letters.

14. As to its symmetry.

15. As to the sum of its coefficients.

16. As to the degree of its highest and lowest terms.

17. As to its homogeneity.

18. As to the number of terms.

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19. Write down the most useful type-forms.

20. State what arrangement of terms is best in multiplication; in division.

21. Explain cross multiplication; and show how it is used in multiplying numerals.

22. Explain the use of detached coefficients, in multiplication; in division.

23. Show how the symmetry of the factors helps to determine the product; the quotient.

24. Explain the methods of contraction in multiplication; in division.

25. Explain the checks used in multiplication; in division.

26. Explain synthetic division.

27. Draw a line whose length equals the sum of two given lines, and show by a diagram that the square on this line is made up of a square on each of the two given lines and two rectangles having these lines as sides: hence illustrate the formula $(a+b)^2 = a^2 + 2ab + b^2$.

So, the formulæ $(a-b)^2 = a^2 - 2ab + b^2$, $(a+b)\cdot(a-b) = a^2 - b^2$.

28. How might the knowing that the product of homogeneous factors is homogeneous help to find errors in division?

Add, and arrange the sum to falling powers of x and the coefficients to falling powers of y.

29. $x^4 + 4xy^3 - 4xz^3 + 4x^3y - 4x^3z + 6x^2y^2 + 6x^2z^2 - 12xy^2z$
$+ 12xyz^2 - 12x^2yz + y^4 - 4y^3z - 4yz^3 + z^4 + 6y^2z^2 + z^4$.

Expand and add:

30. $(a+b+c)^2 + (a+b-c)^2 + (a-b+c)^2 + (-a+b+c)^2$.

31. $(a+b+c)^3 + (a+b-c)^3 + (a-b+c)^3 + (-a+b+c)^3$.

32. $(a+b+c)^4 + (a+b-c)^4 + (a-b+c)^4 + (-a+b+c)^4$.

Given $\alpha + \beta = -b/a$, $\alpha\beta = c/a$; then:

33. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = b^2/a^2 - 2c/a = (b^2 - 2ca)/a^2$.

34. $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = (-b^3 + 3abc)/a^3$.

35. $\alpha^4\beta^7 + \alpha^7\beta^4 = -(b^2 - 3ac)\cdot bc^4/a^7$.

36. $1/\alpha + 1/\beta = -b/c$. 37. $\alpha/\beta + \beta/\alpha = (b^2 - 2ca)/ca$.

38. $1/\alpha^2 + 1/\beta^2 = (b^2 - 2ca)/c^2$.

39. $(\alpha/\beta - \beta/\alpha)^2 = b^2(b^2 - 4ac)/a^2c^2$.

III. SIMPLE EQUATIONS.

An equation is a statement that two expressions are equal. These two expressions are the *members* of the equation.

An *identity* is an equation that is true for every value of the letters involved; the sign of identity is $\equiv$.

E.g., $(x+a)\times(x-a)\equiv x^2-a^2$ for all values of x and a;

but the equation $5x+2=17$ is true only when x is 3,

and the equation $x^2-3x=4$ is true only when x is 4 or -1.

The letter or letters whose values are sought are the *unknown elements;* the other elements are *known elements.* The unknown elements of an equation are usually represented by the last letters of the alphabet; and in literal equations the first letters then stand for known elements.

The *solution* of an equation, or set of equations, consists in making such transformations as shall result in giving the values of the unknown elements. The values so found are the *roots* of the equation or set of equations; and the test to be applied to them is to replace the unknown elements by these values, and see whether they make the equations true.

E.g., of the equation $2x=4$, [x unknown], 2 is a root,
$\because 2\cdot 2=4.$ [df. root.

So, of the equation $x^2-5x+6=0$, [x unknown], 2, 3 are roots,
$\because 2^2-5\cdot 2+6=0,\quad 3^2-5\cdot 3+6=0.$

The degree of an equation is that of its highest term.

If the unknown element enter an equation by its first power only, the equation is a *simple equation.*

E.g., $2x=4$, [x unknown] is a simple equation.

AXIOMS OF EQUALITY.

1. *Numbers equal to the same number are equal to each other.*
2. *If to equal numbers equals be added, the* SUMS *are equal.*
3. *If from equal numbers equals be subtracted, the* REMAINDERS *are equal.*

4. *If equal numbers be multiplied by equals, the* PRODUCTS *are equal.*

5. *If equal numbers be divided by equals, the* QUOTIENTS *are equal.*

6. *If equal numbers be raised to like integer powers, the* POWERS *are equal.*

7. *If of equal numbers like roots be taken, the* ROOTS *are equal.*

QUESTIONS.

1. Show that 7 is not a root of the equation $x-4=2$.

What number is a root of this equation?

2. Write an equation that is not simple.

3. Show that $^-3$ is a root of the equation $x^2=9$.

What other root has this equation? are the two roots equal?

4. What is the difference between an axiom and a theorem?

5. If from each member of the equation $x+5=11-2x$, 5 be subtracted, what term in the first member is cancelled?

What change is made in the second member?

So, if $2x$ be added, what change is made in each member?

6. May the equation $x+a=b-2x$ be written $x+2x=b-a$?

How, then, may a term be transposed from one member of an equation to the other, and by authority of what axioms?

7. In the equation $\frac{2}{3}x=\frac{1}{2}x+1$, by what single number may the two fraction-terms be multiplied so as to become integers?

What other term must then be multiplied by the same number, and for what reason?

8. What axiom gives authority for changing the signs of all the terms of an equation?

9. When from the equation $5x=12$, we get the value $2\frac{2}{5}$ for x, what axiom is applied?

So, when from the equation $\sqrt{x}=3$, $x=9$ is found?

10. Write an equation of the second degree with x, y as unknown elements; so, of the third degree.

§ 1. ONE UNKNOWN ELEMENT.

PROB. 1. TO SOLVE A SIMPLE EQUATION, ONE UNKNOWN ELEMENT.

Multiply both members of the equation by some number that contains as factors all of the denominators, if any; [ax. 4.

transpose to one member all terms that involve the unknown element and to the other member all other terms; [axs. 2, 3.

reduce both members to their simplest form;

divide both members by the coefficient of the unknown element. [ax. 5.

CHECK. *In the original equation, replace the unknown element by the result so found.*

E.g., if $\frac{1}{6}(x+12)=\frac{1}{7}(6+3x)-\frac{1}{6}x$; [$x$ unkn.

then $\because 7x+84=36+18x-7x$, [mult. by 42.

and $7x-18x+7x=36-84$, [trans. 84, $18x$, $-7x$.

$\therefore -4x=-48$, and $x=12$; [div. by -4.

and $\because \frac{1}{6}(12+12)\equiv\frac{1}{7}(6+36)-2$, [repl. x by 12.

$\therefore$ 12 is the root sought.

If the known elements be wholly or in part literal the process is essentially the same.

E.g., if $ax-(bx+1)/x=a(x^2-1)/x$,

then $ax^2-bx-1=ax^2-a$, [mult. by x.

$ax^2-ax^2-bx=-a+1$, [trans. ax^2, -1

$bx=a-1$, [cancel ax^2, div. by $^-1$.

$x=(a-1)/b$. [div. by b

So, if $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0$,

then $ax-bx-ac+bc-bx+cx+ab-ac-cx+ax+bc-ab=0$,

$2ax-2bx=2ac-2bc$,

$x=c$. [div. by $2(a-b)$.

QUESTIONS.

Find values of x that make true equations of the statements:

1. $12-5x=13-x$. 2. $1-5x=7x+3$. 3. $3x+6-2x=7x$.

4. $8+4x=12x-16$. 5. $a-2x=x-b$. 6. $m-nx=px+q$.

7. $2x+\frac{1}{3}(4+x)=3\frac{2}{3}$. 8. $4x-1=\frac{1}{2}x+2x+\frac{1}{8}$.

9. $(x+4)(x-2)=(x-9)(x-3)$. 10. $(x+1)(x-1)=x(x-2)$.

11. $(x-2)(x-7)+(x+1)(x-3)-8x=2(x-8)(x-7)-2$.

12. $\frac{1}{x+1}+\frac{2}{x+2}=\frac{3}{x+3}$. 13. $\frac{36x}{x-2}=44$. 14. $\frac{2x-3}{3x+4}=\frac{6x+5}{9x-10}$.

15. $\frac{x-a}{2x-b}=\frac{3x-c}{6x-d}$. 16. $\frac{3x-5}{2}=\frac{6x+5}{7}$. 17. $\frac{1}{x}-\frac{3}{2x}+\frac{5}{7x}=\frac{3}{28}$.

18. $\frac{1}{9}(7x+5)-\frac{1}{3}(5-x)=7\frac{1}{3}-\frac{1}{2}x-\frac{1}{6}(8-7x)$.

19. $\frac{1}{3}(x+10)-\frac{3}{5}(3x-4)+\frac{1}{6}(3x-2)(2x-3)=x^2-\frac{8}{15}$.

20. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$. [simp. each mem. separately.

21. $\frac{1}{x-3}-\frac{1}{x-4}=\frac{1}{x-5}-\frac{1}{x-6}$. 22. $\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}$.

23. $\frac{1}{2}(x-\frac{1}{3}a)+\frac{1}{3}(x-\frac{1}{4}a)+\frac{1}{4}(x-\frac{1}{5}a)=0$.

24. $24-x-[7x-\{8x-(9x-\overline{3x-6x})\}]=0$.

25. $(x+5)^2=(4-x)^2+21x$. 26. $(x-m)(x+n)=x(x-q)$.

27. $\frac{x-1}{a}-\frac{x-2}{b}=\frac{2}{b}$. 28. $\frac{ax^2}{b-cx}+a+\frac{ax}{c}=0$.

29. $ax-b(x-1)-c=0$. 30. $(a^2-x)(a^2+x)=a^4+2ax-x^2$.

31. $(x+1)^2=x[6-(1-x)]-2$. 32. $(x-1)^2=(x-2)(x+3)$.

33. $(5x-6)/m-(x-1)/n=x-2$. 34. $px-qx=(p+q)x-q^2$.

35. $\frac{1}{3}(x+4)-\frac{1}{5}(x-4)=2+\frac{1}{15}(3x-1)$.

36. $\frac{1}{4}(3x+2)-\frac{1}{3}(12-x)=\frac{1}{2}x$. 37. $(x+a)(x-b)=(x-c)(x+d)$.

38. $ax-m-2\{bx-n-3[cx-p-4(dx-q)]\}=0$.

39. $x-[2x+(3x-4x)]-5x-\{6x-[(7x+8x)-9x]\}=-30$.

40. $\frac{1}{3}(x-3)(x+2)-\frac{1}{6}(x-4)(2x+1)=7$.

SPECIAL PROBLEMS.

Note 1. If the statement of the problem be in words, that statement must be first translated into algebraic form.

E.g., to divide \$6341 among A, B, C, so that B shall have \$420 more than A, and C \$560 more than B:

put x for A's share, $x+420$ for B's, $x+420+560$ for C's;

then $\because x+\overline{x+420}+\overline{x+420+560}=6341,$

$\therefore 3x=6341-420-420-560=4941,$ and $x=1647;$

$\therefore$ A has \$1647, B has \$2067, C has \$2627.

So, to divide 144 into four parts, such that the first part increased by 5, the second decreased by 5, the third multiplied by 5, and the fourth divided by 5, are all equal:

put x for the number to which the several results are equal;

then $\because \overline{x-5}+\overline{x+5}+\overline{x:5}+\overline{x\cdot 5}=144,$

$\therefore 5x-25+5x+25+x+25x=720,$ [mult. by 5.

i.e., $36x=720,$ $x=20,$

and the parts are $20-5,\ 20+5,\ 20:5,\ 20\cdot 5$; i.e., 15, 25, 4, 100.

So, to find a number such that if 5, 15, 35 be added to it, in turn, the product of the first and third sums shall be 10 more than the square of the second:

put x for the number, $x+5,\ x+15,\ x+35$ for the three sums;

then $\because (x+5)\times(x+35)=(x+15)^2+10,$

$\therefore x^2+40x+175=x^2+30x+225+10,$

$\therefore 10x=60,$ $x=6,$ and the numbers are 11, 21, 41.

So, the width of a room is two thirds of its length; if the width were three feet more and the length three feet less, the room would be square; find its dimensions:

put x for a side of the supposed square;

then the length of the room is $x+3$ and the width $x-3,$

and $\because x-3=\frac{2}{3}(x+3),$

$\therefore 3x-9=2x+6,$ $x=15,$ and the room is 12 by 18 feet.

In solving problems, it is not sufficient that the result found shall satisfy the equation: it must also satisfy the conditions of the problem as expressed in words.

QUESTIONS.

1. If to the double of a certain number 14 be added, the sum is 154: what is the number?

2. If to a certain number 46 be added, the sum is three times the original number: find the number.

3. The sum of two numbers is 20, and if three times the smaller number be added to five times the larger, the sum is 84: what are the numbers?

4. Divide 46 into two parts such that if one part be divided by 7 and the other by 3, the sum of the quotients shall be 10.

5. In a company of 266 men, women, and children, there are four times as many men and twice as many women as children: how many men are there? how many women? and how many children?

6. Thirty yards of cloth and forty yards of silk together cost $66; the silk is worth twice as much per yard as the cloth: find the cost per yard of each of them.

7. My purse and the money it contains are together worth $20, and the purse is worth a seventh part of the money: how much money does the purse contain?

8. A shepherd being asked how many sheep he had in his flock, said "if I had as many more, half as many more, and 7 sheep and a half, I should then have 500": how many sheep had he?

9. A is 58 years older than B, and A's age is as much above 60 as B's age is below 50: find their ages.

10. What number is that whose double being added to 24, the sum is as much above 80 as the number itself is below 100?

11. What number is that from which if 5 be subtracted, two thirds of the remainder is 40?

12. A and B together can do a piece of work in 8 days, A and C in 9 days, B and C in 10 days: in how many days can each man do the work alone? in how many days can they do it all working together?

* GENERAL FORMS.

NOTE 2. Every simple equation with one unknown element may be reduced to the form $ax+b=a'x+b'$, whose solution gives $x=(b'-b)/(a-a')$, a value that may be positive, negative, zero, infinite, or indeterminate, according to the relations between the elements a, a', b, b'; and there are three cases:

(*a*) $a \neq a'$; then x has a single value, positive, negative, or zero, that satisfies the equation.

(*b*) $a=a', b \neq b'$; then $x=\infty$, wherein ∞, read *infinity*, denotes a number larger than can be named.

This result may be interpreted by saying that if a and a', or either of them, take gradually changing values, and if a be not equal to a' but approach nearer and nearer to a', then x grows larger and larger without bounds.

E.g., if A, A′ travel along the same road in the same direction at a, a' miles an hour, and if A be now b miles and A′, b' miles from the same starting point; then the quotient $(b'-b)/(a-a')$ is the time that will elapse before they come together.

If the hourly gain, $a-a'$, be small, that time is long;

if there be no gain, i.e., if $a=a'$, they will never be together, and there is no finite value of x that satisfies the equation.

(*c*) $a=a', b=b'$; then $x=0/0$, and the equation is satisfied by any number whatever.

These cases may be further illustrated by this question:

Two men A, A′ have b, b' dollars and save a, a' dollars a year: in how many years will they have the same assets? The interpretation of the principles in terms of the problem is this:

If $b'>b$, the time sought is in the future if $a>a'$, but in the past if $a<a'$,

and if $b'<b$, these results are reversed;

if $b \neq b', a'=a$, the answer is *never*;

if $b=b'$, the present is the time sought if $a \neq a'$, but if $a=a'$, the two men have always the same assets.

QUESTIONS.

1. If the equation $x-20=-3x$ be represented by the general formula $ax+b=a'x+b'$, for what number does each of the letters a, b, a', b', stand?

2. In the fraction $(b'-b)/(a-a')$ what is the sign of the denominator if $a>a'$? if $a<a'$?

So, if $b'>b$, what is the sign of the numerator? if $b'<b$?

3. What is known about the value of x if $b'>b$ and $a>a'$? So, if $b'>b$ and $a<a'$? if $b'<b$ and $a>a'$? if $b'=b$?

4. What is the value of a fraction whose numerator is zero? Show that this value multiplied by the denominator gives the numerator, and that no other value will give it.

5. Reduce the fractions 6/3, 6/.3, 6/.03, 6/.003 ··· to whole numbers: what change is going on in the series of denominators and what in the quotients? if the denominator be very small, what is the quotient? if the denominator be 0?

6. How is an example in division proved?

Prove that: $0/0=2$; $0/0=10$; $0/0=5000$; $0/0=-.12$.

What is the value of $(b'-b)/(a-a')$ when $b'=b$, $a=a'$?

What is meant by an indeterminate expression?

7. In the case of the two travelers A, A'; if $a=a'$, $b=b'$, are they now together? how long have they been together? how long will they remain together?

8. If $b'>b$, $a<a'$, is the time of meeting past or future? if $b'>b$, $a>a'$? if $b'<b$, $a<a'$? if $b'<b$, $a>a'$?

9. What is the meaning of the problem if b, b' be of opposite signs? both negative? if a, a' be of opposite signs? both negative?

10. A gives a house worth b dollars and land worth a dollars an acre in exchange for B's house worth b' dollars and as many acres of land worth a' dollars an acre: how large is each estate?

Discuss the problem for the different relations between a, a', b, b' considered before: which of the results interpreted in the last question on page 76 has no meaning here?

§ 2. TWO UNKNOWN ELEMENTS.

Equations that involve the same unknown elements, and are satisfied by the same values of them, are *simultaneous equations;* and those values are *simultaneous values.*

E.g., if the equations $2x+5y=19$, $6x-3y=3$, [x, y unknown] be simultaneous, 2, 3 are a pair of roots,

$\because 2\cdot 2+5\cdot 3=19, \quad 6\cdot 2-3\cdot 3=3.$

Elimination is that process by which an unknown element is removed from a pair of equations.

PROB. 2. TO ELIMINATE AN UNKNOWN ELEMENT FROM A PAIR OF SIMPLE EQUATIONS.

BY ADDITION AND SUBTRACTION.

Find some number, as small as may be, that exactly contains both the coefficients of the element to be eliminated;

divide this number, in turn, by these coefficients, and multiply the two equations through by the quotients; [ax. 4.

subtract one equation from the other, member from member.

E.g., to eliminate x from the pair of equations

$6x+7y=85, \quad 2x+3y=33:$

then $\because$ 6 contains 6 once and 2 three times,

$\therefore 6x+7y=85, \quad 6x+9y=99,$ [mult. by 1, 3.

$\therefore 2y=14.$ [subtract.

BY COMPARISON.

Solve both equations for the element to be eliminated; [pr. 1.

put the two values thus found equal to each other. [ax. 1.

E.g., to eliminate x from the same pair of equations:

then $x=\frac{1}{6}(85-7y)=\frac{1}{2}(33-3y)$, [sol. both eq. for x.

BY SUBSTITUTION.

Solve either equation for the element to be eliminated; [pr. 1.

in the other equation, replace this element by the value so found.

E.g., to eliminate x from the same pair of equations:

then $\because x=\frac{1}{2}(33-3y)$, [sol. 2d eq. for x.

$\therefore 99-9y+7y=85.$ [repl. x in 1st eq.

QUESTIONS.

1. Define elimination; what is the derivation of the word?

2. In the equation $2x+5y=19$ replace x by $-\frac{1}{2}$, y by 4: is the equation true for these values?

Is the equation $6x-3y=3$ true for the same values?

So, in the second equation replace x by 3, y by 5: do these values satisfy the first equation?

3. Assume any value at random for y in $2x+5y=19$: can a satisfactory value be found for x in that equation? in $6x-3y=3$? with the same value of y, in both equations at the same time? what is needful to a correct solution?

4. What two axioms are applied in elimination by addition and subtraction?

5. Multiply the equation $6x+7y=85$ by 3, $2x+3y=33$ by 7; then, subtracting, what letter is eliminated?

6. In eliminating x by comparison, how is it known that the two expressions for x, found from the separate equations, are equal?

7. To get definite values for two unknown elements, how many independent equations must be used?

8. By addition and subtraction, eliminate x from the pair of simultaneous equations $5x+6y=29$, $3x+2y=11$.

9. So, from $2x+5y=23$, $7x+2y=34$.

10. Eliminate y from $8x+13y=79$, $7x+2y=41$.

11. By comparison, eliminate x from the pair of simultaneous equations $4x-3y=-10$, $7x+8y=62$.

12. So, from $\frac{3}{2}x+4y=18$, $5x-3y=17$.

13. Eliminate y from $2x+4y=20$, $7x+3y=37$.

14. So, from $4/x+7/y=1\frac{2}{3}$, $3/x+5/y=1\frac{1}{4}$, using $1/x$, $1/y$ as the two unknown elements.

15. By substitution eliminate x from the pair of simultaneous equations $3x-2y=1$, $5x+3y=3\frac{1}{4}$.

16. So, from $6x+9y=15$, $8x-15y=11$.

17. Eliminate y from $\frac{1}{5}x-\frac{5}{7}y=-4$, $3x+4y=43$.

THE SOLUTION OF SIMULTANEOUS SIMPLE EQUATIONS.

Prob. 3. To solve a pair of simple equations, if one has two unknown elements and the other but one.

Solve the equation that has but one unknown element; [pr. 1.
replace this element by its value in the other equation, and solve for the other unknown element. [th. 4 cr. 2.

E.g., to find x, y from the pair of simultaneous equations
$6x+7y=85,\quad 4x=24$:

then $x=6,\quad 36+7y=85,\quad 7y=49,\quad y=7.$

Prob. 4. To solve a pair of simple equations, if both have the same two unknown elements.

Combine the two equations so as to eliminate one unknown element, thus forming an equation involving the other;
solve this equation for its unknown element;
replace this element by its value in either of the given equations;
solve the equation so found for the other unknown element.

Check. *Replace the two unknown elements by their values in that one of the original equations which was not used in finding the value of the second element.*

E.g., to find x, y from the pair of simultaneous equations
$6x+7y=85,\quad 2x+3y=33$:

then $\because \frac{1}{7}(85-6x)=\frac{1}{3}(33-2x)$, [elim. y.
$\therefore 255-18x=231-14x$, [mult. by 21.
$\therefore -4x=-24,\quad x=6;$
$\therefore 36+7y=85,\quad y=7.$

DEPENDENT EQUATIONS.

Note 1. If one of the two equations may be derived from the other, there is no single solution, but an infinite number of solutions. The equation is then *indeterminate.*

E.g., the equations $2x+3y=13$, $6x+9y=39$ are but one equation in two forms, and any value may be given to either of the unknown elements, and the corresponding value of the other computed.

QUESTIONS.

Solve the pairs of equations below, and check the work:

1. $5x-3y=15,\quad 2y=10.$
2. $3(x+2y)=30,\quad \frac{3}{2}x=3.$
3. $8x+3y=14,\quad 5y=10.$
4. $3x-8y=7,\quad 3\frac{1}{2}x=5.$
5. $2x-4-y=x-1,\quad -3y=-9.$
6. $2\frac{1}{2}+y-\frac{3}{2}x=\frac{1}{2}y,\quad \frac{7}{4}x=3\frac{1}{2}.$
7. $x+y=9,\quad x-y=1.$
8. $15x+2y=17,\quad 9x-4y=5.$
9. $5x+3y=8,\quad 7x-3y=4.$
10. $3x+y=16,\quad 3y+x=8.$
11. $3y=5x,\quad 16y=27x-1.$
12. $8x=5y,\quad 13x=8y+1.$
13. $x=\frac{2}{5}y,\quad x-\frac{1}{2}y=\frac{2}{3}.$
14. $11x-3y=0,\quad x-y=-16.$
15. $2x+y=0,\quad \frac{1}{2}y-3x=8.$
16. $x-y=\frac{5}{6},\quad x+1=\frac{3}{2}.$
17. $\frac{8}{x}-\frac{5}{y}=\frac{1}{6},\quad \frac{7}{x}-\frac{3}{y}=\frac{5}{6}.$
18. $\frac{3}{x}+\frac{2}{y}=1\frac{3}{12},\quad \frac{5}{x}+\frac{7}{y}=\frac{29}{12}.$
19. $\frac{2}{x}+\frac{3}{y}=\frac{7}{12},\quad \frac{2}{x}-\frac{3}{y}=-\frac{1}{12}.$
20. $\frac{5}{x}-\frac{3}{y}=-\frac{1}{6},\quad \frac{3}{x}-\frac{1}{y}=\frac{1}{30}.$
21. $210x+42y+93=0,\quad 22x+14y+7=0.$
22. $\frac{3}{8}y-\frac{1}{5}x+24=0,\quad \frac{2}{7}y+\frac{1}{3}x+11=0.$
23. $\frac{1}{3}(\frac{1}{4}x-\frac{1}{5}y+\frac{1}{6})=\frac{1}{4}(x-y),\quad \frac{1}{2}(\frac{1}{4}y-\frac{1}{5}x+\frac{1}{3})=\frac{1}{4}(x+y).$
24. $\frac{x+y}{x-2y}=3,\quad \frac{x-3y}{6}+\frac{5y-x}{9}=\frac{1}{2}.$
25. $\frac{1}{15}(80+3x)=18\frac{1}{3}-\frac{1}{7}(4x+3y-8),$
    $10y+\frac{1}{5}(6x-35)=55+10x.$
26. $\frac{12y+7x}{5x-3y}=1,\quad \frac{18y-7x}{6x+10}=2.$
27. $\frac{x}{a+b}+\frac{y}{a-b}=2a,\quad \frac{x-y}{4ab}=1.$
28. $1-\frac{x+y}{x-y}=\frac{3x}{x-y},\quad \frac{7x-3y}{23}=3.$
29. $\frac{1}{2(x+1)}+\frac{4}{3(y+1)}=5,\quad \frac{1}{x+1}-\frac{1}{3(y+1)}=1.$

30. Write down any simple equation at will, and then make dependent equations from it by different processes.

SPECIAL PROBLEMS.

NOTE 2. In solving special problems it may be convenient to express different unknown elements by different symbols.

E.g., a vintner at one time sells 20 dozen of port wine and 30 dozen of sherry, and for the whole receives \$600; and at another time he sells 30 dozen of port and 25 dozen of sherry, at the same price as before, and for the whole receives \$700: what are the prices?

put x for the price of a dozen of port, and y for that of a dozen of sherry;

then $\because 20x+30y=\$600, \quad 30x+25y=\$700,$

$\therefore x=\$15, \quad y=\$10.$

So, if a certain rectangular bowling-green were 5 yards longer and 4 yards broader, it would contain 113 yards more; but if 4 yards longer and 5 yards broader, it would contain 116 yards more: what are its length and breadth?

put x, y for the length and breadth;

then $\because (x+5)\cdot(y+4)=xy+113, \quad (x+4)\cdot(y+5)=xy+116,$

$\therefore x=12$ yds., $y=9$ yds.

So, if the number of men engaged upon a certain piece of work be made 5 greater, the work can be done in 4 days; if 5 less, in 12 days: how many men are at the work, and in how many days can they do it?

put x for the number of men and y for the number of days;

then one man could do the work in xy days,

and $\because 4(x+5)=xy, \quad 12(x-5)=xy,$

$\therefore 4(x+5)=12(x-5), \quad x=10, \quad y=6.$

So, a certain two-figure number is 6 greater than 6 times the sum of its digits, and reversing the order of the digits makes the number less by 3 times its first figure; find the number:

put x for the tens' figure and y for the units' figure;

then $\because 10x+y=6(x+y)+6, \quad 10y+x=10x+y-3x,$

$\therefore x=9, \quad y=6,$ and the number is 96.

QUESTIONS.

1. Find two numbers such that their sum is 27, and that, if four times the first be added to three times the other, the sum is 93.

2. Find two numbers such that twice the first and three times the second together make 189 and, if double the second be taken from five times the first, 7 remains.

3. A flagstaff is sunk in the ground one-sixth part of its height, the flag occupies 6 feet, and the rest of the staff is three-quarters of its whole length: what is the length?

4. The diameter of a five-franc piece is 37 millimeters and that of a two-franc piece 27 millimeters; thirty pieces laid in contact in a straight line measure one meter: how many of each kind are there?

5. A certain number consisting of two figures is equal to four times the sum of its digits, and if 18 be added to it the order of the digits is reversed: what is the number?

6. If the tail of a fish weigh 9 lbs., his head as much as his tail and half his body, and his body as much as his head and tail, what is the weight of the whole fish?

7. There are two pipes one of which will fill a cistern in an hour and a half, the other in two hours and a quarter: in what time will both fill it?

8. Divide 90 into four parts such that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient so found shall all be equal.

9. A and B engage in play; in the first game A wins as much as he had and $4 more and finds he has twice as much as B; in the second game B wins half as much as he had at first and $1 more, when he has three times as much as A: what sum had each at first?

10. What fraction is that which becomes equal to $\frac{3}{4}$ when its numerator is increased by 6, and equal to $\frac{1}{2}$ when its denominator is diminished by 2?

DISCUSSION OF A PROBLEM.

NOTE 3. To *discuss* a problem whose answer is numerical is to try whether all the conditions of the problem are satisfied by all or any of the numbers that are found to satisfy the equations into which the problem was translated; and, if not, to observe what other conditions the unknown elements must satisfy besides those taken account of in putting the problem into equation.

To discuss a problem whose answer is literal is to observe between what limiting numerical values of the known elements the problem is possible; whether any singularities or remarkable circumstances occur within these limits; and what changes in the statement of the problem would make it possible for the other values of the known elements.

E.g., in a certain two-digit number the first digit is half the other, and if 27 be added to the number, the order of the digits is reversed: what is the number?

put x for first digit, y for second digit;
then $\because 2x=y$, $10x+y+27=10y+x$,
$\therefore x=3$, $y=6$, the number is 36; and $36+27=63$.

Were this the statement: in a certain two-digit number, the first digit is half the other, and if 24 be added to the number, the order of the digits is reversed;
then $\because 2x=y$, $10x+y+24=10y+x$,
$\therefore x=2\frac{2}{3}$, $y=5\frac{1}{3}$, and the number is impossible.

The statement of the problem puts a limitation upon x, y not expressed by the equation: they must be integers.

And were this the statement: in a certain two-digit number the first digit is half the other, and if a be added to the number, the order of the digits is reversed;
then $2x=y$, $10x+y+a=10y+x$, $x=a/9$, $y=2a/9$.

Here the special condition is imposed that a shall be a multiple of 9 not greater than 36 nor less than -36;
i.e., a is 36, 27, 18, 9, 0, $^-9$, $^-18$, $^-27$, $^-36$,
and the number is 48, 36, 24, 12, 0, $^-12$, $^-24$, $^-36$, $^-48$.

QUESTIONS.

1. In a company of a persons each man gave m dollars to the poor, each woman n dollars; the whole sum was ka dollars: how many men were there? how many women?

Show that, if $m>n$, then $m>k>n$; and that the example is possible only when $(m-k)a$, $(k-n)a$ are multiples of $m-n$.

2. A is a years old and B b years: when will A be twice as old as B? What relation between a and b puts the date sought in the future? what, in the present? what, in the past?

3. A laborer receives a dollars a day when he works, and forfeits b dollars a day when idle; at the end of m days he receives k dollars: how many days does he work, and how many is he idle?

What relation exists between a, b, k, m, if his forfeits just cancel his earnings? if his forfeits exceed his earnings? Give numerical illustrations.

4. A father is now a times as old as his son; k years hence he will be b times as old: what are their ages now?

Give numerical values to a, b, k, and interpret the results.

Show that: $k>0$, if $a>b$; $k=0$, if $a=b$; $k<0$, if $a<b$.

5. The sum of two numbers is a, and the difference of their squares is k^2; what are the numbers?

Interpret the results: if $k^2>a^2$; if $k^2=a^2$; if $k^2<a^2$.

6. The difference of two numbers is a, and the difference of their squares is k^2: what are the numbers?

Interpret the results: if $k^2>a^2$; if $k^2=a^2$; if $k^2<a^2$.

7. If to the numerator of a certain simple fraction a be added, the result is c/d, and if to the denominator a' be added, the result is c'/d': what is the original fraction?

8. In a certain two-digit number the second digit is a times the first, and if b be added to the number, the digits are reversed: show that a may not exceed 9, be less than 1, or be negative; and show when a may be fractional.

Show that b is a multiple of 9 or of $a-1$; and show what bounds b lies between for different values of a.

MORE CONDITIONS THAN UNKNOWN ELEMENTS.

NOTE 4. It may happen that the problem gives more conditions, and so more equations, than unknown elements; such problems can be solved if the conditions be not inconsistent.

E.g., to find x, y from the set of three equations

$$3x+7y=17, \quad 5x-2y=1, \quad 8x+y=10:$$

take the first two equations and solve;

then $\because x=1, \quad y=2,$ in these two equations,

and $\because$ these roots satisfy the third equation,

$\therefore$ this set of equations is possible, and the roots are 1, 2.

But not possible is the set of equations

$$3x+7y=17, \quad 5x-2y=1, \quad 8x+y=12.$$

FEWER CONDITIONS THAN UNKNOWN ELEMENTS.

NOTE 5. It may happen that the problem gives fewer conditions, and so fewer equations, than unknown elements; such problems are *indeterminate*, and the set of equations may have many sets of roots.

E.g., to find x, y from the single equation $2x+3y=12$:

put $y=\cdots {}^-4, \ {}^-3, \ {}^-2, \ {}^-1, \ 0, \ {}^+1, \ {}^+2, \ {}^+3, \ {}^+4, \ \cdots,$

then $x=\cdots {}^+12, \ {}^+10\frac{1}{2}, {}^+9, \ {}^+7\frac{1}{2}, \ {}^+6, \ {}^+4\frac{1}{2}, \ {}^+3, \ {}^+1\frac{1}{2}, \ 0, \ \cdots,$

i.e., if to y be given a series of values increasing by 1, there results a series of values for x decreasing by $1\frac{1}{2}$;

or put $x=\cdots {}^-4, \ {}^-3, \ {}^-2, \ {}^-1, \ 0, \ {}^+1, \ {}^+2, \ {}^+3, \ {}^+4, \ \cdots,$

then $y=\cdots {}^+6\frac{2}{3}, \ {}^+6, \ {}^+5\frac{1}{3}, \ {}^+4\frac{2}{3}, \ {}^+4, \ {}^+3\frac{1}{3}, \ {}^+2\frac{2}{3}, \ {}^+2, \ {}^+1\frac{1}{3}, \cdots;$

i.e., if to x be given a series of values increasing by 1, there results a series of values for y decreasing by $\frac{2}{3}$.

If either of the unknown elements take any value whatever, the corresponding value of the others may be found.

E.g., if $x=4\frac{3}{4}$, then $y=\frac{5}{6}$; or if $y=4\frac{1}{2}$, $x=-\frac{3}{4}$.

If the condition be imposed that the roots shall all be integers, or all positive integers, it may happen that the equations have very few such roots, or even none at all.

E.g., 3, 2, is the only pair of positive integer roots in the example above.

QUESTIONS.

1. Given the three simple equations
$2x+3y=18$, $3x-2y=1$, $7x-4y=5$: is it certain, before solving, that values of x and y can be found that will satisfy all the given equations?

Solve the second and third equations and see whether the values so found satisfy the first.

2. So, for the three equations
$x-4y=10$, $4x+10y=14$, $-2x+3y=9$.

3. Find two numbers whose sum is 60, whose difference is 24, and one of which is 3 times the other.

If the results obtained do not satisfy all three conditions, show what change in each condition will make it consistent with the other two.

4. Solve the equations $6x-8y=3$, $15x=7\frac{1}{2}+20y$, or explain what is the difficulty with them.

5. In the example of note 5, why does x decrease when y increases and increase when y decreases?

6. In the equation $4x-5y=1$, if increasing values be given to x, will the corresponding values of y increase or decrease?

If the values of x increase by 1, how do the values of y change? if the values of y decrease by 1, how do the values of x change?

By what integers may x increase or decrease so that y also shall change by integers only?

What are all the pairs of integer roots smaller than 20?

7. Show that the equation $xy=24$ may have an infinite number of pairs of roots, and that the value of one root grows smaller as the other grows larger.

Show how the relations of x, y differ in this example from their relations in ex. 6.

Discuss the equation $xy=0$.

8. How many pairs of roots has the equation $x=ay$?

What relation have the values of x and of y when a is positive? when a is negative?

§ 3. THREE OR MORE UNKNOWN ELEMENTS.

PROB. 5. TO SOLVE A SET OF n INDEPENDENT SIMPLE EQUATIONS THAT INVOLVE THE SAME n UNKNOWN ELEMENTS.

Combine the n equations, two and two, in $n-1$ ways, so that each equation is used at least once, and so as to eliminate the same unknown element at each operation;
thereby form $n-1$ equations involving the same $n-1$ unknown elements;
so, combine these $n-1$ equations, and thereby form $n-2$ equations involving the same $n-2$ unknown elements;
and so on till there results one equation, involving but one unknown element;
solve this equation, and replace the unknown element by its value in one of the two equations involving two unknown elements;
solve this equation for the second unknown element, and replace these two elements by their values in one of the three equations involving three unknown elements;
and so on till all the roots are found.

E.g., to find x, y, z from the set of equations

$$x+2y+3z=14, \quad 3x+2y+z=10, \quad 6x+9y+13z=63:$$

then $6x+12y+18z=84$ [4] [mult. first eq. by 6.

$6x+4y+2z=20$ [5] [mult. sec. eq. by 2.

$6x+9y+13z=63$ [6]

$8y+16z=64$ [7] [sub. eq. 5 from eq. 4.

$5y+11z=43$ [8] [sub. eq. 5 from eq. 6.

$40y+80z=320$ [9] [mult. eq. 7 by 5.

$40y+88z=344$ [10] [mult. eq. 8 by 8.

$8z=24$ and $z=3$ [sub. eq. 9 from eq. 10.

$8y=64-16z=16$ and $y=2$.

$x=14-2y-3z=1$.

QUESTIONS.

1. How many independent equations make it possible to find the value of four unknown elements? of five? of ten? of n?

Solve the sets of equations:

2. $3x-4y+5z=4, \quad 8x-y-z=6, \quad 7x-5y-3z=-1.$
3. $x-2y-5z=20, \quad 3x-5y-3z=22, \quad -8x+11y+9z=-57.$
4. $\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{11}{6}, \quad \frac{2}{x}-\frac{3}{y}+\frac{4}{3z}=\frac{71}{18}, \quad \frac{3}{x}+\frac{4}{y}-\frac{1}{7z}=\frac{20}{21}.$
5. $\frac{1}{3}(y-z)=\frac{1}{2}(y-x)=5z-4x, \quad y+z=2x+1.$
6. $x+2y+3z+4u=20, \quad x+2y+3z-4u=12,$
   $x+2y-3z+4u=8, \quad x-2y+3z+4u=8.$
7. $2x+y+z=16,$
   $x+2y+z=9,$
   $x+y+2z=3.$
8. $cy+bz=a,$
   $az+cx=b,$
   $bx+ay=c.$
9. $x+y=14,$
   $x+z=16,$
   $y+z=18.$
10. $\frac{x-y}{w}=z-2,$
    $\frac{3x+y}{z}=w+8,$
    $x+y=4w,$
    $w-1=z.$
11. $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}=29,$
    $\frac{5}{x}-\frac{3}{y}+\frac{2}{z}=9,$
    $\frac{3}{x}+\frac{4}{y}-\frac{5}{z}=-2.$
12. $\frac{x}{a}+\frac{y}{b}=1,$
    $\frac{x}{a}+\frac{z}{c}=1,$
    $\frac{y}{b}+\frac{z}{c}=1.$
13. $2(x+1)-3(y-1)+z-2=2,$
    $2(x+1)+4(y+1)-5(z-1)=3,$
    $3(2x+2)-2(y-1)+3(z+1)=29.$
14. $x+y+z=9,$
    $x+2y+4z=15,$
    $x+3y+9z=23.$
15. $(x+1)(5y-3)=(7x+1)(2y-3),$
    $(4x-1)(z+1)=(x+1)(2z-1),$
    $(y+3)(z+2)=(3y-6)(3z-1).$
16. $\frac{9}{x}-\frac{2}{y}=\frac{5}{z}-\frac{3}{x}=\frac{7}{y}+\frac{15}{2z}=4.$
17. $\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=r,$
    $\frac{a}{x}-\frac{b}{y}+\frac{c}{z}=s,$
    $-\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=t.$
18. $x+ay+a^2z+a^3u+a^4=0, \quad x+by+b^2z+b^3u+b^4=0,$
    $x+cy+c^2z+c^3u+c^4=0, \quad x+dy+d^2z+d^3u+d^4=0.$
19. $u+v+w+x+y=10, \quad v+w+x+y+z=15,$
    $w+x+y+z+u=13, \quad x+y+z+u+v=11,$
    $y+z+u+v+w=14, \quad z+u+v+w+x=12.$

NOT ALL THE UNKNOWN ELEMENTS INVOLVED IN EVERY EQUATION.

NOTE 1. An unknown element that does not appear in any equation may be considered as already eliminated from it, and the work is shortened by so much; those unknown elements that are in the fewest equations may be eliminated first.

E.g., to find x, y, z, t, u from the set of equations

$$9x-2z+u=41, \quad (1) \qquad 7y-5z-t=12, \quad (2)$$
$$4y-3x+2u=5, \quad (3) \qquad 3y-4u+3t=7, \quad (4)$$
$$7z-5u=11: \quad (5)$$

of these equations, x appears in two, y in three, z in three, u in four, t in two;

equations 1, 3 may be combined to eliminate x, and equations 2, 4 to eliminate t, and there result two new equations, involving y, z, u;

these two equations may be combined to eliminate y, and there results one equation, involving z, u;

this last equation may be combined with equation 5 to eliminate either z or u at pleasure.

PARTICULAR ARTIFICES.

NOTE 2. The equations may have a certain symmetry as to the unknown elements, or functions of them, that permits shorter processes than those of the general rule; sometimes the sum of such unknown elements, or of the functions, may be got first.

E.g., to find x, y, z from the set of equations

$$\frac{1}{x}+\frac{1}{y}=\frac{4}{15}, \quad \frac{1}{y}+\frac{1}{z}=\frac{11}{60}, \quad \frac{1}{z}+\frac{1}{x}=\frac{1}{4};$$

then $\because \dfrac{2}{x}+\dfrac{2}{y}+\dfrac{2}{z}=\dfrac{7}{10}, \quad \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{7}{20},$ [add, div. by 2.

$$\therefore \frac{1}{x}=\frac{7}{20}-\frac{11}{60}=\frac{1}{6}, \quad \frac{1}{y}=\frac{7}{20}-\frac{1}{4}=\frac{1}{10}, \quad \frac{1}{z}=\frac{7}{20}-\frac{4}{15}=\frac{1}{12};$$

$$\therefore x=6, \qquad y=10, \qquad z=12.$$

QUESTIONS.

Solve the systems of equations:

1. $3x-4y+3z+3v-6u=11,$
$3x-5y+2z-4u=11,$
$5z+4u+2v-2x=3,$
$10y-3z+3u-2v=2,$
$6u-3v+4x-2y=6.$

2. $3z+8u=33,$
$7x-2z+3u=17,$
$4y-2z+v=11,$
$4y-3u+2v=9,$
$5y-3x-2u=8.$

3. $x+2y-3z=-1,$
$4x-4y-z=8,$
$3x+8y+2z=-5.$

4. $x+y+z=0,$
$(b+c)x+(c+a)y+(a+b)z=0,$
$bcx+cay+abz=1.$

5. $5x-2(y+z+v)=-1,$
$4z-3(v+x+y)=2,$
$-12y+3(z+v+x)=3,$
$8v-(x+y+z)=-2.$

6. Three cities, A, B, C, are at the corners of a triangle; from A through B to C is 118 miles; from B through C to A, 74 miles; from C through A to B, 92 miles: how far apart are the three cities?

7. The sum of three numbers is 70; the second divided by the first gives 2 for the quotient and 1 for the remainder, and the third divided by the second gives 3 for both quotient and remainder: find the numbers.

8. A, B, C are three towns forming a triangle; a man has to walk from one to the next, ride thence to the next, and drive thence to his starting point; he can walk, ride, and drive a mile in a, b, c minutes respectively; if he start from B, he takes $a+c-b$ hours; if from C, $b+a-c$ hours; if from A, $c+b-a$ hours: find the length of the circuit.

9. A number is expressed by three figures, whose sum is 19; reversing the order of the first two figures diminishes the number by 180, and interchanging the last two increases it by 18: what is the number?

10. A's money in 9 years at 6 % will produce as much interest as B's and C's together in 4 yrs. 6 mos. at 4 %; B's in 8 yrs. at 5 % as much as A's and C's in 3 yrs. 4 mos. at 6 %; C's in 7 yrs. at 3 % \$42 more than A's and B's in 3 yrs. at 4 %: how much money has each man?

*GENERAL FORMS.

NOTE 6. The two equations $ax+by=c$, $a'x+b'y=c'$, are the type-forms of every pair of simple equations that involve the same two unknown elements; their solution gives

$$x=(cb'-c'b)/(ab'-a'b),\quad y=(ac'-a'c)/(ab'-a'b).$$

The values of x, y for a particular pair of equations depend on the values of a, b, c, a', b', c', and an examination of the possible values and relations of these known elements will determine the possible roots of the pair of equations. There are three general cases:

(a) $ab'\neq a'b$; then x, y have single values, positive, negative, or zero, that satisfy both the equations.

(b) $ab'=a'b$, $cb'\neq c'b$; then $ac'\neq a'c$, $x=\infty$, $y=\infty$.

Here $a/a'=b/b'\neq c/c'$; the equations are inconsistent, and they can be satisfied by no finite values of x, y.

The infinite values may be interpreted by saying that if a, a', b, b', any of them, take changing values, and if $ab'\neq a'b$, but ab' approach nearer and nearer to $a'b$, then x, y grow larger and larger without bounds.

(c) $ab'=a'b$, $cb'=c'b$; then $ac'=a'c$, $x=0/0$, $y=0/0$.

Here $a/a'=b/b'=c/c'$; the equations differ by a factor only, and the values sought are indeterminate.

The general forms of simple equations involving three unknown elements are $ax+by+cz=d$, $a'x+b'y+c'z=d'$, $a''x+b''y+c''z=d''$, whose solution gives

$$x=\frac{d(b'c''-b''c')+d'(b''c-bc'')+d''(bc'-b'c)}{a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c)},$$

$$y=\frac{d(a'c''-a''c')+d'(a''c-ac'')+d''(ac'-a'c)}{b(a'c''-a''c')+b'(a''c-ac'')+b''(ac'-a'c)},$$

$$z=\frac{d(a'b''-a''b')+d'(a''b-ab'')+d''(ab'-a'b)}{c(a'b''-a''b')+c'(a''b-ab'')+c''(ab'-a'b)},$$

and all of these denominators have the same value; but the sign of the second is opposite to that of the first and third.

Various relations among the coefficients may be considered:

If d, d', d'' be all zero, the values of x, y, z are zero, unless the denominator is also zero, and then these values are indeterminate, and the given equations are not all independent.

If d, d', d'' be not all zero, but the denominator be zero, the equations are inconsistent.

For if the first equation be multiplied by $b'c''-b''c'$, the second by $b''c-bc''$, the third by $bc'-b'c$, and the results be added,

then the coefficients of y and z vanish identically, and that of x is $a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c)$, i.e., zero, while the second member is not zero.

QUESTIONS.

1. In the pair of equations given in note 6, what relation between the products $a'b$, ab' makes the denominator of the value of x positive? negative? zero?

So, what relation between the products ac', $a'c$ makes the numerator positive? negative? zero?

If $cb'>c'b$ and $ab'<a'b$, is x positive or negative?

2. If the numerator of a fraction be 0, and the denominator not 0, what is the value of the fraction? if the denominator be 0 and the numerator not 0? if both be 0?

3. If $ab'=a'b$ and $cb'\neq c'b$, find the value of the fraction b/b', and so show that $ac'\neq a'c$, and that x, y are both infinite.

4. If $ab'=a'b$ and $cb'=c'b$, show that $ac'=a'c$.

What then is the numerator of the value of x? what the denominator? of the value of y?

Has the fraction 0/0 any definite value?

5. Given the three simple equations

$$ax+by=c, \quad a'x+b'y=c', \quad a''x+b''y=c'',$$

solve the first two, substitute the values of x, y, so found, in the third, and show that

$$a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c)=0,$$

is the equation of condition for the consistency of the three given equations.

§ 4. QUESTIONS FOR REVIEW.

Define and illustrate:

1. An equation; an identity; an axiom.

2. Known elements; unknown elements; the solution of an equation; the roots of an equation.

3. A simple equation involving one unknown element; two unknown elements; three unknown elements.

4. A pair of simultaneous equations; elimination.

State the axiom:

5. Of equality; of addition; of subtraction; of multiplication; of division; of involution; of evolution.

Give the general rule, with reasons and illustrations, for:

6. Solving a simple equation with one unknown element.

7. Eliminating an unknown element from a pair of simultaneous simple equations by addition and subtraction; by comparison; by substitution.

8. Solving a pair of simultaneous simple equations.

9. Solving a set of simple equations involving three or more unknown elements.

Exhibit the general forms, and explain the special cases, for:

10. A simple equation involving one unknown element.

11. A pair of simple equations involving the same two unknown elements.

What difficulty arises:

12. If one of the two equations be dependent on the other?

13. If there be more conditions than unknown elements?

14. If there be fewer conditions than unknown elements?

Explain what is meant:

15. By putting a problem into equation; by solving a problem; by checking the work.

16. By the discussion of a problem whose answer is numerical; of a problem whose answer is literal.

17. The sum of the three figures of a number is 9; the first figure is an eighth of the number made up of the last two figures taken in order, and the last figure is an eighth of the number made up of the first two figures: find the number.

18. A boatman rows down the river 42 miles in 3 hours; returning, he finds the current only two thirds as strong, and it takes him $10\frac{1}{2}$ hours: find how fast he can row in still water, and how fast the river ran at first.

19. At $3\frac{1}{2}$ miles an hour, I can walk from P to Q in a certain time; but at the rate of 3 miles going and 4 miles returning, it takes me five minutes longer: how long is the round trip?

20. A crew that can pull 9 miles an hour in still water takes twice as long to come up a river as to go down it: find the velocity of the current.

21. If a rectangle be made 3 feet longer and 3 feet broader, the area is 102 square feet greater; but if it be shortened 5 feet, and widened 1 foot, the area is 16 square feet less: find the length and breadth.

22. If a concert room contained 10 more benches, one person less might sit on a bench; if it contained 15 fewer, 2 more persons must sit on a bench: how many benches are there and how many people on a bench?

23. The perimeter of a rectangular field is 70 rods; if the width were increased one rod and the length diminished two rods, the width would be eight ninths of the length: find the area of the field.

24. The contents of a vessel is 60 per cent alcohol and the rest water; after drawing out 10 gallons of the mixture and filling up the vessel with water, $42\frac{6}{7}$ per cent of the contents is alcohol: find the capacity of the vessel.

25. A cask contains 18 gallons of wine and 12 of water, and another contains 3 gallons of wine and 9 of water: how many gallons must be drawn from each cask to make the mixture contain 7 gallons of wine and 7 of water?

26. Every year a merchant adds 40 per cent to his capital, but takes out $3000 for expenses; at the end of the third year, after deducting his $3000, he finds that he has doubled his original capital and has $1800 besides: how much had he at first? how much at the end of each year?

27. A lady receives $2160 yearly interest on her capital, but if it were loaned at a half of one per cent higher, she would receive $240 more: find her capital and the rate of interest.

28. How much pure copper must be added to 35 pounds of silver, 15 parts pure out of 16, so that the mixture shall contain 4 parts of pure silver to one part of alloy?

29. In a naval action, a third of a fleet was taken, a sixth was sunk and two ships were burnt; afterwards a seventh of the remaining ships were lost in a storm, and only 24 ships were left: how large was the fleet at first?

30. A and B begin a game, both having the same sum of money, and the loser is always to pay the winner a dollar more than half what the loser has; after B has lost one game and won one, he has twice as much money as A: how much had he at first?

31. A besieged garrison had bread enough to last 6 weeks, giving each man 10 oz. a day; but, after they lost 1200 men in a sally, the allowance was increased to 12 oz. a day, and the bread lasted 8 weeks: how many men were there at first?

32. A man and his family use a barrel of flour in m days, but when the man is away it lasts n days longer: how long would it last the man alone?

33. A man was engaged to work 48 days at two dollars a day and his board, which was estimated at a dollar a day; at the end of the time he received $42, his employer having deducted the cost of board for the days he was idle: how many days had he worked?

34. Out of a certain sum of money, a man paid his creditors $432, lent a friend a third of the remainder, and spent a quarter of what was then left; after this he had a fifth of the original sum: how much money had he at first?

PROBLEMS OF PURSUIT.

35. A is 12 miles behind B, and gains 2 miles an hour; when will they be together? When, if A have m miles to gain, and gain h miles in k hours?

36. A is 180 miles east of B, A's train runs 25 miles an hour and B's 20 miles: when will they be together if A goes west and B east? A east and B west? both west? both east? when, if A be m miles east of B and they travel at a, b miles an hour?

Interpret the results if m be positive, negative; a, positive, negative; b, positive, negative; $a>b$; $a=b$; $a<b$.

37. B has a start of h hours, A goes c miles an hour faster than B, and overtakes him in k hours: what is each man's rate?

Interpret the results if c be negative; h negative; k negative.

38. A, B starting together walk around a track; at the end of half an hour A has made 3 circuits and B $4\frac{1}{2}$: when will B next pass A? when, if A has made a circuits and B, b circuits?

39. The circle of a clock face is divided into 12 hour-arcs: how many arcs does the hour hand pass over in an hour? in two hours? ··· in twelve hours? the minute hand?

How many arcs does the minute hand gain, over the hour hand, in one hour? in two hours? ··· in twelve hours?

40. How many hour-arcs has the minute hand gained, over the hour hand, when they are together between one and two? between two and three? ··· between ten and eleven?

41. At what time are the hour hand and minute hand opposite to each other between twelve and one? between one and two? ··· between five and seven? ··· between eleven and twelve? At what times between twelve and twelve are the two hands at right angles to each other?

42. If A, B, C starting together at noon walk around a track, A ten times in an hour, B twelve times, and C fifteen times, when will A and B be together for the first time? B and C? C and A? When will they be all together?

When will A be midway between B and C? B midway between C and A? C midway between A and B?

43. If the hour, minute, and second hands of a clock all turn upon the same centre, at what times will the minute and second hands be together? at what times will each of the three hands be midway between the other two?

44. A passenger train 200 ft. long passes a freight train 680 ft. long in 30 seconds when they are running in opposite directions, and in one minute when in the same direction: find the rate of each train.

PERCENTAGE AND SIMPLE INTEREST.

45. Put b for the basis of percentage, p for the percentage, r for the rate, a for the amount of the basis and the percentage, v for the net value of the basis less the percentage; then, by definition, $r=p/b$, $a=b+p$, $v=b-p$.

From these three fundamental equations show that:

$$b=p/r,\quad p=b\cdot r,\quad a=b\cdot(1+r),\quad b=a/(1+r),$$
$$v=b(1-r),\quad b=v/(1-r).$$

Translate these six formulæ into theorems and into rules.

46. Put p for the principal, r for the yearly rate, t for the time in years, i for the simple interest, a for the amount; then, by definition, $i=p\cdot r\cdot t$, $a=p+i$.

From these two fundamental equations, show that

$$p=i/rt,\ r=i/pt,\ t=i/pr,\ a=p\cdot(1+rt),\ p=a/(1+rt).$$

Translate these five formulæ into theorems and into rules.

47. *Bank discount* is simple interest prepaid, and the *bank present worth*, or *proceeds*, is the principal less the discount.

Put v for the present worth, then, by definition, $v=p-i$.

Show that $v=p\cdot(1-rt)$, $p=v/(1-rt)$.

Translate these two formulæ into theorems and into rules.

Show the relation between the true present worth and the bank present worth for a given principal, rate, and time; and that between the true rate earned and the bank rate.

AVERAGES.

48. If the foreman gets \$5 a day, two sub-foremen \$3 each, and forty men \$2 each, what is the average daily wages for the whole party?

Replace the numerals by letters and make a general formula.

49. A grocer mixes 20 lbs. of tea worth 50 cents a pound with 30 lbs. worth 40 cents, and 50 lbs. worth 30 cents, and sells the whole at 45 cents; what is the real value? the profit on one pound? the rate of profit? Make a general formula.

50. With debts $p_1, p_2, p_3, \cdots$ due at times $t_1, t_2, t_3 \cdots$ from a fixed date, find a single date when the whole may be paid without loss to debtor or creditor.

Discuss the problem if some of the t's or p's be negative.

WORK.

51. A man can do a piece of work in n days: what part of it can he do in one day? in two days? in three days? in ten days? in n days? in $2n$ days?

52. A man can do n units of work in one day: in what part of a day can he do one unit? two units? three units? ten units? n units? $2n$ units?

53. A can do a piece of work in a days, B in b days, C in c days: what part of the work can A and B together do in one day? B and C? C and A? A, B and C?

54. A can do a units of work in a day, B, b units, C, c units: in what part of a day can A and B together do a unit of work? B and C? C and A? A, B and C?

55. A can do a units of work in a' days, B, b units in b' days, C, c units in c' days: in how many days can they together do $a+b+c$ units?

56. To do a certain piece of work A needs m times as long as B and C together; B, n times as long as C and A; C, p times as long as A and B: what relation have m, n, p?

57. A reservoir is filled by pipes A, B in c hours, by pipes B, C in a hours, by pipes C, A in b hours: in what time is it filled by each pipe running alone? by the three pipes together?

58. A reservoir holding m gallons is filled by two pipes, running a, b gallons an hour, and emptied by two pipes running c, d gallons an hour: what is the relation between a, b, c, d so that, with all the pipes running, the reservoir, if empty, shall be filled in h hours? if full, emptied in h hours?

IV. MEASURES AND MULTIPLES.

§ I. INTEGERS.

The product of two integers is a *multiple* of either integer; and either integer is a *measure* of the product.

E.g., $^+15$, $^-15$, are multiples of $^+5$, $^-5$, $^+3$, $^-3$;
and $^+5$, $^-5$, $^+3$, $^-3$, are measures of $^+15$, and of $^-15$.

Every integer is a multiple of itself, its opposite, and $^\pm1$.

COMMON MULTIPLES AND MEASURES.

If the same number be a multiple of two or more integers, it is a *common multiple* of them; and a measure of two or more integers is a *common measure* of them.

E.g., $^+30$, $^-30$, are common multiples of 1, $^-5$, 10, $^-15$, 30;
and $^+3$, $^-3$, are common measures of 3, $^-6$, 9, $^-15$, 30.

The smallest of all the common multiples of two or more integers is their *lowest common multiple;* and the largest of all their common measures is their *highest common* measure.

E.g., $^\pm30$ is the lowest common multiple of 3, 10, $^-15$, but not of 3, $^-15$;
and $^\pm3$ is the highest common measure of 6, $^-9$, 12, but not of 6, 12.

Ax. 1. *The sum, the difference, and the product of two integers are integers.*

Theor. 1. *A common measure of two or more integers is a measure of their sum.*

Let A, B, $\cdots$ be any integers and M a common measure of them; then is M a measure of the sum $A+B+\cdots$

For $\because A=M\cdot a$, $B=M\cdot b,\cdots$, and $a, b,\cdots$ are integers, [hyp.

$\therefore A+B+\cdots=M\cdot a+M\cdot b+\cdots$

$=M\cdot(a+b+\cdots)$; [I, th. 7.

and $\because$ the sum $a+b+\cdots$ is an integer, [ax. 1.

$\therefore$ M is a measure of $A+B+\cdots$ Q. E. D. [df. msr.

COR. *A common measure of two integers is a measure of the sum and of the difference of any multiples of them.*

QUESTIONS.

1. Explain the difference between a product and a multiple; between a divisor and a measure.

2. Name some multiples of 6, of $^-3$, of 0, of 1.

3. Name some measures of 24, of 60, of $^-64$, of 1, of 0.

4. Name some common multiples of 2, 3, $^-4$, 1; of 5, 7, 11.

5. Name some common measures of 24, 64, 120.

6. Name the lowest common multiple of 6, 15, 10: what is their highest common multiple?

7. Name the highest common measure of $^-120$, $^-45$, $^-60$: what is their lowest common measure? What common measure of two of these numbers is not a measure of the third?

8. In the proof of theor. 1, why must a, b, $\cdots$ be integers? Explain the dependence of the last statement of the proof on that just before it.

9. Prove that a common measure of two integers is a measure of their difference and of their product.

10. Prove that a measure of the sum of two or more integers, if it measure all of them but one, measures that one also.

11. Prove that a measure of an integer is a measure of any multiple of that integer; that a multiple of an integer is a multiple of all measures of that integer.

12. What two common measures have all integers? What other two measures has any given integer?

13. Why is any positive integer a multiple of its opposite? Why is any measure of a negative integer a measure of the same integer with the positive sign?

14. In finding the highest common measure of integers, will the answer be affected if all the integers be taken positive?

15. Prove that the sum, the difference, and the product of two even numbers are even.

EUCLID'S PROCESS FOR FINDING THE HIGHEST COMMON MEASURE.

THEOR. 2. *If the larger of two integers be divided by the smaller, the common measures of the divisor and the remainder are the common measures of the two integers.*

If the smaller integer be divided by the remainder, this divisor by the second remainder, and so on, then some remainder is zero.

The last divisor is the highest common measure of the two integers.

Let A, B be two integers, Q_1 the quotient of A by B, and R_1, $R_2, \cdots R_n$ the successive remainders;

then $\because R_1 = A - B \cdot Q_1$, [df. rem.

$\therefore$ the common measures of A, B are measures of R_1. [th. 1, cr.

i.e., the common measures of A, B are common measures of B, R_1.

So, $\because A = B \cdot Q_1 + R_1$,

$\therefore$ the common measures of B, R_1 are measures of A,

i.e., the common measures of B, R_1 are common measures of A, B;

$\therefore$ the common measures of A, B are the common measures of B, R_1, and there are no others. Q.E.D.

2. $\because R_1, R_2 \cdots$ are all integers, and grow smaller and smaller,

$\therefore$ some one of them, say R_n, is 0. Q.E.D.

3. $\because$ the common measures of B, R_1 are the common measures of R_1, R_2, [above

$\therefore$ the common measures of A, B are the common measures of R_1, R_2; and so on;

$\therefore$ the common measures of A, B are the common measures of R_{n-2}, R_{n-1};

and $\because R_{n-1}$ is the highest common measure of R_{n-2}, R_{n-1},

$\therefore R_{n-1}$ is the highest common measure of A, B. Q.E.D.

If R_{n-1} be 1, then A, B are usually said to have no common measure; for unity, being a measure of all integers, is not a characteristic common measure of A, B.

COR. *Every remainder,* $R_1, R_2 \cdots R_n$ *is the difference of two multiples of* A, B.

E.g., $R_1 = A - B \cdot Q_1,$

$$R_2 = B - R_1 \cdot Q_2 = B - (A - B \cdot Q_1) \cdot Q_2 = -A \cdot Q_2 + B \cdot (1 + Q_1 \cdot Q_2),$$
$$R_3 = R_1 - R_2 \cdot Q_3 = A \cdot (1 + Q_2 \cdot Q_3) - B \cdot (Q_1 + Q_3 + Q_1 \cdot Q_2 \cdot Q_3).$$

QUESTIONS.

1. In division, how does the remainder compare, in size, with the divisor? if the divisor be divided by the remainder, how does the second remainder compare with the first?

In Euclid's process how do the remainders change?

Can any remainder be a fraction or a negative number?

Why must an exact divisor be reached at last?

2. If R_n be the last remainder, for what two things does R_{n-1} stand? for what three things does R_{n-2} stand?

3. Show that the following statements are true:

any measure of B and R_1 is a measure of A,
any measure of R_1 and R_2 is a measure of B,
. . . .
any measure of R_{n-2} and R_{n-1} is a measure of R_{n-3},
any measure of R_{n-1} and R_n is a measure of R_{n-2}.

4. What measure of R_{n-2} is found by Euclid's process?

Is this measure also a measure of R_{n-1}?

Of what other successive pairs of numbers is R_{n-1} a measure?

5. What is the highest measure of R_{n-1}? the highest common measure of R_{n-1}, R_{n-2}? of R_{n-2}, R_{n-3}? $\cdots$ of A, B?

6. Find the highest common measure of 14637, 51306.

7. Draw any two unequal lines and call them a, b; lay off b on a as many times as it can be repeated; lay off the remainder, c, on b; lay off the remainder, d, on c, and so on, till there is no remainder: what is true of the last line used?

8. Prove that every common measure of two integers is the difference of two multiples of the integers.

PRIME NUMBERS.

An integer that has no measures but itself and unity is a *prime number;* and two integers that have no common measure except unity are *prime to each other.*

E.g., 2, $^-3$, 5, $^-7$, 11 are prime numbers; and 9, $^-25$ are prime to each other, though not prime numbers.

An integer that can be measured by another integer is a *composite number.*

E.g., 15, $^-35$, whose prime measures are 3, 5; 5, 7.

THEOR. 3. *If two integers be prime to each other, then two multiples of them can be found such that their difference is unity; and conversely.*

For, let A, B be two integers that are prime to each other, and $R_1, R_2, \cdots, R_n$ the remainders as in theor. 2;

then $\because$ A, B are prime to each other, [hyp.

$\therefore R_{n-1}$, their highest common measure, is 1;

and $\because R_{n-1}$ is the difference of two multiples of A, B, say $m \cdot A$, $n \cdot B$, [th. 2 cr.

$\therefore m \cdot A - n \cdot B = 1.$ Q.E.D.

Conversely, let $m \cdot A - n \cdot B = 1$;

then $\because$ every common measure of A, B is a measure of 1,

$\therefore$ A, B are prime to each other. Q.E.D.

THEOR. 4. *If an integer be prime to two or more integers, it is prime to their product.*

Let A, B, C, $\cdots$ be any integers and P an integer that is prime to each of them;

then is P prime to the product $A \cdot B \cdot C \cdots$

For take m, n; p, q; r, s; $\cdots$ integers such that

$m \cdot P - n \cdot A = 1$, $p \cdot P - q \cdot B = 1$, $r \cdot P - s \cdot C = 1, \cdots$, [th. 3.

then $\because (m \cdot P - n \cdot A) \times (p \cdot P - q \cdot B) \times (r \cdot P - s \cdot C) \cdots = 1$,

i.e., $h \cdot P + k \cdot A \cdot B \cdot C \cdots = 1$, wherein $h \cdot P$ is the sum of all the terms that contain P, and $k = \pm n \cdot q \cdot s \cdots$

$\therefore$ P is prime to the product $A \cdot B \cdot C \cdots$ Q.E.D. [th. 3 conv.

Cor. 1. *If two integers be prime to each other, so are their positive integer powers.*

QUESTIONS.

1. Name seven prime integers; the only even prime integer.

2. If all the integers from 1 to 1000 be written in order, and all the even numbers be struck out, every third number from 3, every fifth from 5, and so on, what numbers remain?

3. Name five integers, no one of which is prime, but which are all prime to each other.

4. Is it possible for an integer to have but one factor?

Separate 36 into sets of two factors, of which one shall be 2, 3, 4, 6, 9, 12, 18, 36, in turn: were all the possible sets obtained before the entire series of divisors were tried?

In trying all possible factors of an integer in the order of their size, how far need the process be carried?

5. If two integers that are prime to each other be subjected to Euclid's process, what is the last remainder? the last divisor? the last remainder but one?

6. Find two multiples of 9, 17, whose difference is 1; so, of 5, 13; of 11, 13; of 13, 17; of 17, 19; of 13, 19.

7. What is the converse of a theorem?

State the converse of theor. 3 as a separate theorem.

8. In the converse of theor. 3, why have A, B no common measure but 1?

9. What statement in theor. 4 is a direct application of theor. 3? what, of the converse of that theorem?

10. Are all prime numbers prime to each other?

If an integer be resolved into its prime factors, what relation have other prime numbers to these factors separately?

What relation have these numbers to the original number?

11. Prove that an even number can not measure an odd number; that a number having any even factor is even; and that the product of any number of odd factors is an odd number. Is the sum of two odd numbers odd or even?

COR. 2. *If an integer measure the product of two integers and be prime to one of them, it measures the other.*

Let A, B be two integers and P an integer that measures the product $A \cdot B$ and is prime to A; then P measures B.

For, let $m \cdot A$, $n \cdot P$ be two multiples of A, P such that
$mA - nP = 1$; [th. 3.

then $(m \cdot A - n \cdot P) \cdot B = m \cdot A \cdot B - n \cdot P \cdot B = B$,

and ∵ P measures both $A \cdot B$, and $P \cdot B$, [hyp., df. msr.

∴ P measures B. Q.E.D. [th. 1 cr.

COR. 3. *If a prime number measure the product of two or more integers, it measures at least one of them.*

COR. 4. *If a prime number measure a positive integer power of an integer, it measures the integer; and if the integer, then the power.*

COR. 5. *If an integer be measured by two integers that are prime to each other, it is measured by their product.*

For, let A be an integer and P, Q integers prime to each other that measure A; and let $A = B \cdot P$;

then ∵ Q measures the product $B \cdot P$ and is prime to P, [hyp.
∴ Q measures B, [cr. 2.

i.e., $m \cdot Q = B$, wherein m is some integer; [df. msr.

∴ $m \cdot P \cdot Q = B \cdot P$, and the product $P \cdot Q$ measures the product $B \cdot P$,

i.e., the product $P \cdot Q$ measures A. Q.E.D.

THEOR. 5. *An integer can be resolved into prime factors in but one way.*

Let the integer N be the product $A^a \cdot B^b \cdot C^c \cdots$ wherein A, B, C, $\cdots$ are primes, and $a, b, c, \cdots$ are positive integers;

then N has no other prime factor.

For ∵ any other prime, F, is prime to each of the prime factors A, B, C, $\cdots$, [hyp.

and to their powers $A^a, B^b, C^c, \cdots$, [th. 4 cr. 1.

∴ F is prime to the product N. Q.E.D. [th. 4.

And A can be a factor of N not more than a times, B not more than b times, and so on.

For $\because$ $N = A^a \cdot B^b \cdot C^c \cdots$,

$\therefore$ $N/A^a = B^b \cdot C^c \cdots$;

and $\because$ A is prime to $B^b \cdot C^c \cdots$, [th. 4.

$\therefore$ A is a factor of N not more than a times.

So, A can be a factor of N not fewer than a times; and so with B, with C, and with the other prime factors. Q.E.D.

COR. *A common measure of two or more integers can contain no factor that is not in all of them.*

QUESTIONS.

1. In the proof of theor. 4 cor. 2, are m, n given numbers? Are they numbers assumed at random?

Why does P measure mAB? why does P measure B?

2. So, if P measure AB, what kind of number is AB/P?

If P be prime to A, is A/P an integer? what then is B/P?

3. If an even number be measured by an odd number, the quotient is even.

4. In the proof of cor. 3, if P measure the product $A \cdot B \cdot C \cdot D \cdots$ and be prime to A, of what is P a measure?

So, if P be prime to A, B, C, D, of what is P a measure?

5. Cor. 4 is found by inserting the word *equal* in cor. 3.

6. A composite number may measure the product of two or more integers and not measure either of them separately.

7. If A be measured by the integers P, Q, R, prime to each other, and if $A = m \cdot P \cdot Q$, then R measures m, and the product $P \cdot Q \cdot R$ measures A.

What application is here made of theor. 4?

8. In the proof of theor. 5, if P be prime to A, why is it prime to A^a? why prime to N? If the factors of N be all different, what are the values of $a, b, \cdots$?

9. If two integers have several common prime factors, the product of these factors is a common measure of the integers.

What factors are found in their highest common measure?

§ 2. ENTIRE FUNCTIONS OF ONE LETTER.

An algebraic expression whose value depends on the value of a single letter is a *function* of that letter; the letter is the *letter of arrangement.*

E.g., the value of the expression x^3-2x^2+3x+5 depends on the value of x, and it is known when x is known.

If the value of an expression depend on the values of two or more letters the expression is a function of those letters.

E.g., the value of $x^3-2x^2y^{-1}+3xy^{-2}+5y^{-3}$ is known when the values of x, y are known.

The letter f stands for the word function.

E.g., fx means a function of x, and fa is what fx becomes when x is replaced by a.

If there be more than one function of x, such functions may be distinguished as fx, $\text{F}x$, $\phi x, \cdots$ or $f_1x, f_2x, f_3x, \cdots$

An *entire function of one letter* is the sum of positive integer powers of that letter, with or without coefficients.

E.g., x^3+3x^2+3x+1 is an entire function of x of the third degree, whose coefficients are integers.

So, $\frac{1}{3}y^3-y^2+y-\frac{1}{3}$ is an entire function of y.

So, if n be a positive integer, $ax^n+bx^{n-1}+cx^{n-2}\cdots+kx+l$ is an entire function of x of the nth degree, with literal coefficients, which may be either integers or fractions.

In general, the definitions and principles established for integers apply with slight changes to entire functions of one letter, and, for the most part, they are here stated, proved, and illustrated in the same words.

The product of an entire function of one letter by another such function is a *multiple* of either function, as to that letter, and either function is a *measure* of the product.

E.g., a^3-3a^2+3a-1 is a multiple of a^2-2a+1, $a-1$, and a^2-2a+1, $a-1$ are measures of a^3-3a^2+3a-1.

So, $8(x-a)(y-b)$ is a measure of $2m(x^2-a^2)(y+b)$ as to a, m, x,
and a multiple of it as to the numerals;
but neither measure nor multiple as to y, nor as to b.

Every entire function of one letter is both a multiple and a measure of itself, and a multiple of all expressions that are free from this letter; and every such expression is a measure of any entire function.

QUESTIONS.

1. Write an entire function of y of the 4th degree, whose coefficients are all negative integers.

So, an entire function of x^2, and one of $x+y$.

2. If n be a fraction, is $nx^5-(n-1)x^4+(n-2)x^3-(n-3)x^2$ an entire function of x?

If n be a fraction and m an integer, is

$nx^m+2nx^{m-1}+\frac{1}{3}nx^{m-2}+\cdots$ an entire function of x?

$x^n+amx^{n-1}+bm^2x^{n-2}$? $a+bx^m+cx^n+dx^{m+n}+cx^{mn}$?

3. If n be an integer, on what two conditions is

$ax^n+bx^{(n-1)/2}+cx^{(n-3)/2}+dx^{(n-5)/2}$ an entire function of x?

$x^n-ax^{n-1}+bx^{n-2}+cx^{n-3}$? $x^n+ax^{(n-2)/2}+bx^{(n-4)/2}+cx^{(n-6)/2}$?

4. If $fx=x^2+3x+6$, write an expression for fa, $f(-y)$; and find the values of $f\,1, f\,0, f\,{}^-2, f\,5, f\,{}^-5, f.167$.

5. When functions of a single letter are under discussion and an expression is said to be free from the letter of arrangement, what do you know about the nature of that expression? Can you tell whether it is an integer or a fraction? whether it is positive or negative? whether it involve powers? or roots?

6. If an entire function of a single letter be divided by any expression free from the letter of arrangement, how do the exponents of the letter of arrangement in the several terms of the quotient compare with those of the given function?

Why, then, is every such expression a measure of every such entire function?

What integer is, for a like reason, a measure of all integers?

COMMON MULTIPLES AND MEASURES.

If the same entire function be a multiple of two or more entire functions, it is a *common multiple* of them; if a measure of them all, it is their *common measure*.

E.g., $7a^4(y^{12}-1)$, $\frac{1}{3}(y^6-1)^2$ are common multiples of
$y+1$, $y-1$, y^2+1, y^2-1, y^2-y+1, y^2+y+1
and $x-a$ is a common measure of x^8-a^8, x^6-a^6, x^4-a^4.

That common multiple of two or more entire functions which is of the lowest degree is their *lowest common multiple;* and that common measure which is of highest degree is their *highest common measure.*

E.g., y^6-1 is the lowest common multiple of
$y+1, y-1, y^2-1, y^2-y+1, y^2+y+1, y^3+1, y^3-1,$
and x^2-a^2 is the highest common measure of x^8-a^8, x^6-a^6, x^4-a^4, x^2-a^2; but not of x^8-a^8, x^4-a^4.

AX. 2. *The sum, the difference, and the product of two entire functions of a letter are entire functions of that letter.*

THEOR. 6. *A common measure of two or more entire functions of a letter is a measure of their sum.*

For, let $ax^n+bx^{n-1}+\cdots+kx+l$, $a'x^m+b'x^{m-1}+\cdots+k'x+l'$, be two entire functions of x, and M a common measure of them;

then $\because$ $ax^n+bx^{n-1}+\cdots+kx+l=\text{M}\cdot\text{P}$,
$a'x^m+b'x^{m-1}+\cdots+k'x+l'=\text{M}\cdot\text{Q}$,
wherein P, Q, are entire functions of x, [hyp.

$\therefore$ $ax^n+bx^{n-1}+\cdots+kx+l+a'x^m+b'x^{m-1}+\cdots+k'x+l'$
$=\text{M}\cdot\text{P}+\text{M}\cdot\text{Q}=\text{M}\cdot(\text{P}+\text{Q})$;

and $\because$ the sum $\text{P}+\text{Q}$ is an entire function of x, [ax. 2.

$\therefore$ M is a measure of
$ax^n+bx^{n-1}+\cdots+kx+l+a'x^m+b'x^{m-1}+\cdots+k'x+l'$,
and so for three or more functions. Q.E.D. [df. msr.

If the entire functions $ax^n+bx^{n-1}+\cdots+kx+l$, $a'x^m+b'x^{m-1}+\cdots+k'x+l',\cdots$ be represented by A, B, $\cdots$, the proof of theor. 1 applies word for word to theor. 6.

COR. *A common measure of two entire functions of a letter is a measure of the sum and the difference of their multiples.*

QUESTIONS.

1. A common multiple of two or more entire functions of one letter is of at least what degree?

A common measure cannot exceed what degree?

2. In what three ways can two entire functions of a letter be combined with absolute certainty that all the exponents of the result will be positive integers?

3. What is the lowest common multiple of $a+b/c$, $a-b/c$, as entire functions of a? as entire functions of b?

4. Following the line of argument of theor. 6, show that a^2-x^2 is a measure of $(a^8-x^8)+(a^6-x^6)+(a^4-x^4)$.

What is P? Q? R? What is the letter of arrangement?

5. Represent a^8-x^8 by A, a^6-x^6 by B, a^4-x^4 by C, and write out the proof of theor. 6.

6. In the proof of theor. 6, if M be a function of the nth degree, what is P?

if M be of $(n-2)$nd degree, of what degree is P?

if $n>m$, what is the highest degree possible to M?

if Q be free from x, what is the degree of M?

if P, Q be both free from x, what relation have m, n?

What is the degree of P+Q?

7. A common measure of two or more entire functions of a letter is a measure of their sum, their difference, and their product: is it also a measure of their quotient?

Illustrate these principles by aid of the functions in ex. 3.

8. A multiple of a multiple of a number is a multiple of the number; a measure of a measure of a number is a measure of the number; and a common measure of two or more numbers is a common measure of any multiples of them.

9. A common multiple of two or more numbers is not necessarily a multiple of their sum; but the sum of two such common multiples is a common multiple of the numbers.

EUCLID'S PROCESS FOR FINDING THE HIGHEST COMMON MEASURE.

THEOR. 7. *If the higher of two entire functions of a letter be divided by the lower, the common measures of the divisor and the remainder are the common measures of the functions.*

If the lower function be divided by the remainder, this divisor by the second remainder, and so on, then some remainder is zero.

The last divisor is the highest common measure of the two entire functions.

Let A, B be two entire functions of a letter, Q_1 the quotient of A by B, and $R_1, R_2, \cdots R_n$ the successive remainders;

then $\because R_1 = A - B \cdot Q_1$, [df. rem.

$\therefore$ the common measures of A, B are measures of R_1; [th. 6 cr.

i.e., the common measures of A, B are measures of B, R_1.

So, $\because A = B \cdot Q_1 + R_1$,

$\therefore$ the common measures of B, R_1 are measures of A;

i.e., the common measures of B, R_1 are measures of A, B;

$\therefore$ the common measures of A, B are the common measures of B, R_1, and there are no others. Q.E.D.

And $\because R_1, R_2, \cdots$ are all entire functions of one letter, and of lower and lower degree,

$\therefore$ some one of them is of zero degree, as to that letter, and, if not itself zero, the next remainder, R_n, is zero.

Q.E.D.

And $\because$ the common measures of B, R_1 are the common measures of R_1, R_2, [above.

$\therefore$ the common measures of A, B are the common measures of R_1, R_2, and so on;

$\therefore$ the common measures of A, B are the common measures of R_{n-2}, R_{n-1};

and $\because R_{n-1}$ is the highest common measure of R_{n-2}, R_{n-1},

$\therefore R_{n-1}$ is the highest common measure of A, B. Q.E.D.

If R_{n-1} be free from the letter of arrangement, then A, B are usually said to have no common measure; for expressions free from the letter of arrangement, being measures of all entire functions of that letter, are not characteristic common measures of A, B.

COR. *Every remainder,* $R_1, R_2, \cdots R_n$ *is the difference of two multiples of the entire functions,* A, B.

QUESTIONS.

1. In the proof of theor. 7, if the higher function be of the mth degree and the lower of the nth, of what degree is the first quotient? the product of the divisor by this quotient?

What is the highest degree possible to the first remainder? to the second quotient? to the second remainder?

Why can no remainder have a negative exponent?

2. The common measures of B, R_1 are common measures, and the only common measures, of R_1, R_2: state the proof.

3. If two entire functions of a letter have an integer as a common measure, is this measure a common factor of all the coefficients?

4. What common measure free from x have the functions
$$8ax^4-16ax^3+24ax^2-12ax-4a,$$
$$24ax^4-44ax^3+64ax^2-24ax-20a\,?$$
Divide out this common factor and find by Euclid's process the highest common measure of the quotients.

5. If all common monomial factors of the two functions have been divided out before applying Euclid's method, what does a remainder free from the letter of arrangement show about the original functions?

6. Show that $A=B\cdot Q_1+R_1$, $B=R_1\cdot Q_2+R_2$, $R_1=R_2\cdot Q_3+R_3$.
Counting R_1, R_2, R_3 as unknown elements, solve these equations for R_3, and show that its value is the difference of two multiples of A, B.

Can all other remainders be expressed as such differences?

7. In ex. 4 express R_1, R_2, R_3 as the differences of two multiples of the given functions.

PRIME FUNCTIONS.

An entire function of a letter that has no measures but itself, and expressions that are free from that letter, is a *prime function;* and two entire functions of a letter that have no common measure except expressions that are free from that letter are *prime to each other.*

An entire function of a letter that has another entire function of that letter as a measure is a *composite function,* and the measuring functions are its factors.

E.g., x^2+x+1, $4x^2-7$ are prime functions of x,

and x^3+x^2-2x is prime to each of them, though itself composite, having the factors x, $x+2$, $x-1$.

THEOR. 8. *If two entire functions of a letter be prime to each other, then two multiples of them can be found such that their difference is free from that letter, and conversely.*

For, let A, B be two functions of a letter that are prime to each other, and R_{n-1} be their highest common measure;

then $\because R_{n-1}$ is the difference of two multiples of A, B, say mA, nB, and is free from the letter of arrangement,

$\therefore m\cdot A-n\cdot B$ is free from the letter of arrangement.

Conversely: let $m\cdot A-n\cdot B$ be free from the letter of arrangement, and not 0;

then $\because$ every common measure of A, B is a measure of an expression that is free from the letter of arrangement,

$\therefore$ A, B are prime to each other. Q.E.D.

THEOR. 9. *If an entire function of a letter be prime to two or more such functions, it is prime to their product.*

Let A, B, C, $\cdots$ be any entire functions of a letter, and P an entire function that is prime to each of them;

then is P prime to the product $A\cdot B\cdot C\cdots$

For, take $m, n, p, q, r, s, \cdots$ entire functions of the letter such that $m\cdot P-n\cdot A$, $p\cdot P-q\cdot B$, $r\cdot P-s\cdot C, \cdots$ are free from the letter of arrangement; [th. 8.

then ∵ the product $(m\cdot\text{P}-n\cdot\text{A})\cdot(p\cdot\text{P}-q\cdot\text{B})\cdot(r\cdot\text{P}-s\cdot\text{C})\cdots$ is free from the letter of arrangement,

i.e., $h\cdot\text{P}+k\cdot\text{A}\cdot\text{B}\cdot\text{C}\cdots$ is free from the letter of arrangement; wherein $h\cdot\text{P}$ is the sum of all the terms that contain P, and $k=\pm n\cdot q\cdot s\cdots$,

∴ P is prime to the product $\text{A}\cdot\text{B}\cdot\text{C}\cdots$ Q.E.D. [th. 8 conv.

QUESTIONS.

1. Are a^3+8, a^3-8 prime functions of a? are they prime to each other? what is the letter of arrangement?

2. What two common measures belong to all integers? how then can integers be called prime to each other?

3. All entire functions have many common measures that are ignored in the definition of functions prime to each other: what are these measures? why are they ignored?

4. In Euclid's process for finding the highest common measure a remainder of unity corresponds, in integers, to an expression free from the letter of arrangement, in entire functions.

5. In the proof of the converse of theor. 8, if the expression $m\cdot\text{A}-n\cdot\text{B}$ be free from the letter of arrangement, every common measure of A, B is free from that letter: state the proof.

6. In the proof of theor. 9, what relation has $h\cdot\text{P}$ to P? $k\cdot\text{A}\cdot\text{B}\cdot\text{C}\cdots$ to the product $\text{A}\cdot\text{B}\cdot\text{C}\cdots$?

How does the last statement depend on the one before it?

7. If there be two entire functions of a letter such that one of them is a measure of the other, an entire function of this letter that is prime to both of them is prime to their quotient.

8. If an entire function of a letter be prime to another such function, it is prime to any integer power of that function.

9. From the fact that a common measure of two numbers measures their sum, prove that 2 is a factor of any number ending in 0, 2, 4, 6, 8; that 4 is a factor, if a factor of the last two figures, and 8, if of the last three figures; and that the remainder got by dividing the last figure of a number by 5 is the same as that for the entire number.

COR. 1. *If two entire functions of a letter be prime to each other, so are their positive integer powers.*

COR. 2. *If an entire function of a letter measure the product of two such functions and be prime to one of them, it measures the other.*

Let A, B be two entire functions of a letter, and P an entire function that measures their product, and is prime to A,

and let mA, nP be multiples of A, P such that $m\text{A}-n\text{P}=l$, wherein l is free from the letter of arrangement; [th. 8.

then $m/l\cdot\text{A}-n/l\cdot\text{P}=1$ and $m/l\cdot\text{A}\cdot\text{B}-n/l\cdot\text{P}\cdot\text{B}=\text{B}$.

and ∵ P measures both A·B and P·B, [hyp., df. msr.

∴ P measures B. Q.E.D. [th. 6 cr.

COR. 3. *If a prime function of a letter measure the product of two or more entire functions of that letter, it measures at least one of them.*

COR. 4. *If a prime function of a letter measure a positive integer power of an entire function of that letter, it measures the function; and if the function, then the power.*

COR. 5. *If an entire function of a letter be measured by two such functions that are prime to each other, it is measured by their product.*

THEOR. 10. *An entire function of a letter can be resolved into factors that are prime functions in but one way.*

Let N, an entire function of one letter, be the product $\text{A}^a\cdot\text{B}^b\cdot\text{C}^c\cdots$ wherein A, B, C, ⋯ are prime functions of that letter and $a, b, c, \cdots$ are positive integers;

then N has no other prime factor.

For ∵ any other prime function, F, is prime to each of the prime factors A, B, C, ⋯ [hyp.

and to their powers $\text{A}^a, \text{B}^b, \text{C}^c, \cdots$ [th. 9 cor. 1.

∴ F is prime to the product N. [th. 9.

So, A can be a factor of N not more, and not fewer, than a times; and so with B, and with the other prime factors.

COR. *A common measure of two or more entire functions of a letter can contain no factor that is not in all of them.*

QUESTIONS.

1. In the proof of theor. 9 cor. 2, if m/l, n/l be fractions, how can B be a multiple of P?

2. Show that cor. 4 is only an application of cor. 3 to factors having a special relation to each other.

3. In cor. 5, let A be an entire function of a letter, and let P, Q be entire functions prime to each other, but measures of A: if $A = n \cdot Q$, what relation has n to P?

If $n = m \cdot P$, what is A in terms of m, P, Q?

4. A measure of an entire function of a letter contains no factor that is not a factor of the function; and a common measure of two or more such functions contains no factor that is not in all the functions.

What are the factors of the highest common measure?

5. If F be a factor of an entire function of a letter, F^n is a factor of the nth power of the function.

6. An entire function of a letter, if measured by any number of such functions that are prime to each other, is measured by their product; and the product of all the prime factors common to two or more such functions is a common measure of the functions.

7. If $a, b, c, \cdots l$ be any entire numbers and m another, if $a', b', c', \cdots l'$ be the remainders when $a, b, c, \cdots l$ are divided by m, and if the sum $a'+b'+c'+\cdots+l'$ be measured by m, so is the sum $a+b+c+\cdots+l$.

If 3 be a factor of the sum of the digits of an integer, it is a factor of the integer; and so with 9.

8. In the last part of the proof of theor. 10, suppose the factor A to occur fewer than a times: what are the only factors by which it can be replaced? Can a power of one of the other factors take the place of a power of A? If A were used more than a times, what change must occur among the other factors? Is that possible?

§ 3. ENTIRE FACTORS.

Integers and entire functions of a letter, or letters. may be classed together as *entire numbers.*

The measures of an entire number are its *entire factors.*

The *prime factors* of a composite entire number are the prime entire numbers whose product is the given number; and *to factor* a number is to find all its prime factors.

E.g., $600a^4x^2-600a^2x^4=2^3\cdot3\cdot5^2\cdot a^2\cdot x^2\cdot(a+x)\cdot(a-x)$.

PROB. 1. TO FACTOR AN INTEGER.

Divide the number, and the successive quotients in order, by the primes 2, 3, 5,$\cdots$, *using each divisor as many times as it measures the successive dividends.*

The successful divisors, and the last undivided dividend, are the prime factors sought; and no divisor larger than the square root of the dividend need be tried.

For the dividend is the product of the divisor and quotient,

and if the divisor be larger than the square root of the dividend, then the quotient is smaller;

i.e., if there be a factor larger than the square root of the dividend, there is also a smaller factor;

and all the possible smaller factors have been already found.

If a number, not a perfect square, have no factor smaller than its square root, it is prime.

E.g., of 11908710, 2 is a successful divisor once, 3 twice, 5 once, 11 once, 23 once, and the square root of the quotient, 523, is smaller than 23;

$\therefore$ the prime factors are 2, 3, 3, 5, 11, 23, 523.

PROB. 2. TO FACTOR A POLYNOMIAL OF KNOWN TYPE-FORM.

Express the number in one of the type-forms, and write its factors directly in the form of the factors of the type.

E.g., $x^2+2ax+a^2-25m^2n^2\equiv(x+a)^2-(5mn)^2$,

$=(x+a+5mn)\cdot(x+a-5mn)$. [II. pr. 3 nt. 4.

QUESTIONS.

1. Make a table of the prime numbers from 0 to 100.

Factor, or prove to be prime:

2. 30; 37; 72; 120; 323; 367; 1331; 1683; 8279; 15625.

3. $a^2+2ab+b^2$.
4. m^2+2m+1.
5. $x^2-2xy+y^2$.
6. x^2+5x+6.
7. $4a^2+2a-20$.
8. m^2-n^2.
9. $4m^2-9x^4$.
10. x^8-y^{16}.
11. $a^{2m}-b^{2n}$.
12. $x^4y^4-16y^4z^4$.
13. x^n-y^n.
14. $e^{2x}-e^{-2x}$.
15. $e^{2x}\pm 2+e^{-2x}$.
16. a^8-256b^{-8}.
17. $a^4x^2+a^3x-a^2$.
18. a^8-c^8.
19. $27c^3+1$.
20. $x^2-13x+42$.
21. $2x^2+6x-8$.
22. x^4-10x^2+9.
23. y^2-y-30.
24. $a^2-4a-32$.
25. a^4-81.
26. $4y^2-2y+1$.
27. $4a^2-4ab+b^2$.
28. $(x\pm y)^2-z^2$.
29. $a^3-a^2x-6ax^2$.
30. $12a^4+a^2x^2-x^4$.
31. $c^2-10cd+25d^2$.
32. $a^{2m}+2a^mb+b^2$.
33. $x^{-2}+6x^{-1}+9$.
34. $x^{-2}-5x^{-1}+6$.
35. $a^{-2m}-2a^{-m}b+b^2$.
36. $9x^2+3ax+6x+2a$.
37. $\overline{x+1}^2+\overline{a+3}\cdot\overline{x+1}+3a$.
38. $x^2+2xy+y^2+5x+5y+6$.
39. $(a-a^{-1})^8-(b-b^{-1})^8$.
40. $(a+x)^6-(a-x)^6$.
41. $a^2x^2-3a^3x+2a^4$.
42. $x^3+y^3+3xy(x+y)$.
43. $a^3-b^3-3ab(a-b)$.
44. $5(x^2-y^2)+3(x+y)^2$.
45. $3(x^2-y^2)-5(x-y)^2$.
46. $2(a^3+a^2b+ab^2)-(a^3-b^3)$.
47. $a^4-b^4+(a^2-b^2)^2$.
48. $2x^3y+5x^2y^2+2xy^3$.
49. $6y^4-3xy^3-9x^2y^2$.
50. $a^2-b^2-c^2-2bc$.
51. $ac+ad+bd+bc$.
52. $x^{2p}+(a+b)x^p+ab$.
53. $x^{2p}-(a+b)x^p+ab$.
54. $x^{2p}+(a-b)x^p-ab$.
55. $x^{2p}-(a-b)x^p-ab$.
56. $x^2+y^2+z^2-2xy\pm 2xz\mp 2yz$. [observe the signs.
57. $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$, [$4a^2b^2-(a^2+b^2-c^2)^2$.
58. $a^2+4b^2+9c^2+\cdots+4ab+6ac+\cdots+12bc+\cdots+\cdots$
59. $m^3-n^3-m(m^2-n^2)+n(m-n)^2+(m^2+n^2)(m-n)$.
60. $a^2-ab-2(ab-b^2)+3(a^2-b^2)-4(a-b)^2$.
61. $(x+y)^2+2(x^2+xy)-3(x^2-y^2)+4(xy+y^2)$.

Prob. 3. To factor an entire function of one letter.

Find the common factors of the coefficients and divide them out; [th. 6 cr.

by trial, or by comparison with known type-forms, find a factor of degree not higher than half the degree of the function.

If all the coefficients be integers, try no factor unless its first and last coefficients measure the first and last coefficients of the function;

try no factor unless its value measures that of the function when the letters have convenient values given to them;

if all the coefficients in the function be positive, try no factor whose first and last coefficients are not both positive.

E.g., of $40ax^2+130axy+75ay^2$ the factors are
5, a, $8x^2+26xy+15y^2$;

and $\because$ 1, 2, 4, 8 are the measures of 8, and 1, 3, 5, 15, of 15, and all the coefficients are positive,

$\therefore$ the possible factors of $8x^2+26xy+15y^2$ are

$x+y$,	$2x+y$,	$4x+y$,	$8x+y$,
$x+3y$,	$2x+3y$,	$4x+3y$,	$8x+3y$,
$x+5y$,	$2x+5y$,	$4x+5y$,	$8x+5y$,
$x+15y$,	$2x+15y$,	$4x+15y$,	$8x+15y$.

In $8x^2+26xy+15y^2$, and in these sixteen possible measures put $x=1$, $y=1$;

then $8x^2+26xy+15y^2=49$, whose measures are 1, 7, 49,

and only $4x+3y, =7$, and $2x+5y, =7$, pass this test;

and, by multiplication, $4x+3y$, $2x+5y$ are found to be the factors sought.

So, of $7x^3-30x^2+62x-45$, the possible linear factors are

$x\pm1$,	$x\pm3$,	$x\pm5$,	$x\pm9$,	$x\pm15$,	$x\pm45$,
$7x\pm1$,	$7x\pm3$,	$7x\pm5$,	$7x\pm9$,	$7x\pm15$,	$7x\pm45$.

In $7x^3-30x^2+62x-45$, and in these factors, put $x=1$;

then the function is -6, and the only possible factors of it are
$x+1$, $x-3$, $x+5$, $7x-1$, $7x-5$, $7x-9$.

So, put $x=2$; then the function is 15, and out of the six possible factors above the only ones still possible are $x+1,\quad x-3,\quad 7x-9$;

and $\because$ of these three factors $7x-9$ succeeds, and the others fail,

$\therefore 7x-9,\quad x^2-3x+5$ are the factors sought.

QUESTIONS.

1. In factoring an entire function of one letter, why need no factor of degree higher than half that of the function be tried? what direction in the rule for factoring integers is like this?

2. Of what two terms is the first term of the dividend the product? the last term of the dividend?

3. Is x^2+x+1 a factor of x^3-1 for certain values only of x or for all values?

In these functions replace x by $1, 2, 3, \cdots$ in turn, and show that the first result measures the second in every case.

4. Let x^3+bx^2+cx+d be an entire function of x, and b, c, d be all positive; divide by $x-a$; show that a positive remainder is left at every step of the process; and that, therefore, a function whose coefficients are all positive has no measure of the form $x-a$.

Factor, or prove to be prime:

5. $6x^4+x^2y-12y^2$.
6. $6b^2x^2-7bx^3-3x^4$.
7. $6x^2+(2a+1)x-(a+2)$.
8. $15x^4+8x^3y-32xy^3-15y^4$.
9. $abx^2+a^2x+b^2x+ab$.
10. $a^3x^3-b^3y^3+c^3z^3$.
11. $a^3x^3+2a^2bx^2+2ab^2x+b^3$.
12. $a^4x^4+a^2b^2x^2+b^4$.
13. $3a^2+6abz-4acz-8bcz^2$.
14. $4x^4-(9b^2+16a^2)x^2+36a^2b^2$.
15. $x^4+(a-b+c-d)x^3+(-ab+ac-ad-bc+bd-cd)x^2$
$\quad +(-abc+abd-acd+bcd)x+abcd$.
16. $aby^3+(a^2+a^2b^2+b^2)y^2+(ab+a^3b+ab^3)y+a^2b^2$.
17. $45x^3+83x^2y-100xy^2-49y^3$.
18. $7x^3-25x^2+11x+3$.
19. $5x^2+17x+3$.
20. $7x^3-10x^2+9x+5$.
21. $a^2\pm ab+b^2$.
22. $a^3\pm a^2b+ab^2\pm b^3$.
23. $18a^3-24a^2-19a+18$.
24. $8x^3-26x^2+29x-12$.
25. $12x^4-16x^3-11x^2-8x-42$.

LINEAR FACTORS.

First degree factors are *linear factors.*

E.g., $x-a$, $y+z$, $z-3+c$.

THEOR. 11. *If fx be an entire function of x, then $x-a$ measures $fx-fa$.*

For, let $fx \equiv A + Bx + Cx^2 + \cdots + Lx^n$, wherein $A, B, C \cdots L$ are free from x, but may contain other letters and numerals;

then $fa = A + Ba + Ca^2 + \cdots + La^n$,

and $fx - fa = B(x-a) + C(x^2-a^2) + \cdots + L(x^n - a^n)$,

which is measured by $x-a$. Q.E.D.

COR. 1. *If fx be divided by $x-a$, the remainder is fa.*

For $fx = B(x-a) + C(x^2-a^2) + \cdots + L(x^n-a^n) + fa$,

and each term of the function in this form is divisible by $x-a$, except fa, which is free from x, and is the remainder. Q.E.D.

COR. 2. *If $fa=0$, then $x-a$ measures fx, and conversely.*

From this corollary comes a new rule for finding linear factors of a function of one letter:

In the function, replace the letter of arrangement by any number; if the function is thereby made zero the letter of arrangement less this number is a factor.

E.g., to factor $x^4 - 8x^3 + 9x^2 + 38x - 40$:

put 1 for x; then $fx = 1 - 8 + 9 + 38 - 40 = 0$, and $x-1$ is a factor, with the quotient $x^3 - 7x^2 + 2x + 40$;

put 1 for x in this quotient; then $f_1x = 1 - 7 + 2 + 40 = 36$, and $x-1$ is not again a factor;

put 2 for x; then $f_1x = 8 - 28 + 4 + 40 = 24$, and $x-2$ is not a factor;

put ${}^-2$ for x; then $f_1x = -8 - 28 - 4 + 40 = 0$, and $x+2$ is a factor, with the quotient $x^2 - 9x + 20$,

whose factors are $x-4$, $x-5$.

$\therefore x^4 - 8x^3 + 9x^2 + 38x - 40 = (x-1) \cdot (x+2) \cdot (x-4) \cdot (x-5)$.

QUESTIONS.

1. If in $x^n - a^n$ x be replaced by a, what does the value of the expression $x^n - a^n$ become? what does this prove?

So, if x be replaced by ^-a, when n is even?

So, if x be replaced by ^-a, when n is odd?

So, if in $x^n + a^n$, x be replaced by ^-a when n is odd?

Is $x^n + a^n$ divisible by either $x-a$ or $x+a$ when n is even?

2. State, as theorems, all the conclusions reached in ex. 1.

3. Divide $x^3 - 6x^2 + 10x - 8$ by $x-2$, and compare the remainder with $f2$.

So, divide by $x-4$ and compare with $f4$.

4. If fx be 0 when x is replaced in turn by $a, b, c, \cdots$, then fx is divisible by $(x-a)\cdot(x-b)\cdot(x-c)\cdots$

5. $x-a$ is a factor of

$$(x^3+2x+3)\cdot(a^3+a)-(a^3+2a+3)\cdot(x^3+x).$$

Find the linear factors of:

6. $6x^3 - 7x^2 - 8x + 16$.

7. $a^4 + 4a^3 + 4a^2 + 4a + 8$.

8. $x^4 + 4x^3 - 25x^2 - 16x + 84$.

9. $x^3 - 8x^2 + 19x - 12$.

10. $a^3 - 7a^2 + 14a - 8$.

11. $y^4 - 4y^3 - 7y^2 + 10y$.

12. $x^3 + 8x^2 + 20x + 16$.

13. $z^3 - 3z^2 + 10z - 8$.

14. $c^4 - 13c^2 + 36$.

15. $y^4 - 11y^2 + 18y - 8$.

16. $x^4 - x^3 - 39x^2 + 24x + 180$.

17. $x^3 + 5x^2 - 9x - 45$.

18. $a^3 - 8a^2 + 13a - 6$.

19. $x^4 - x^3 - 11x^2 + 9x + 18$.

20. $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2$.

21. $3a^2 - 6abz + 4acz - 8bcz^2$.

22. $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48$.

23. $3(a+b)^3 + 8(a+b)^2 + a + b - 2$.

24. $x^3 - (a-b+c)x^2 + (ac-ab-bc)x + abc$.

25. $x^qy^r + y^qz^r + z^qx^r - x^ry^q - y^rz^q - z^rx^q$ is measured by $(x-y)\cdot(y-z)\cdot(z-x)$ if q, r be any positive integers.

By the light of ex. 25, find the factors of:

26. $x^2y + y^2z + z^2x - xy^2 - yz^2 - zx^2$.

27. $x^3y + y^3z + z^3x - xy^3 - yz^3 - zx^3$.

28. $x^3y^2 + y^3z^2 + z^3x^2 - x^2y^3 - y^2z^3 - z^2x^3$.

FRACTIONS IN THEIR LOWEST TERMS.

THEOR. 12. *If the terms of a simple fraction be prime to each other, the fraction can be reduced to no equivalent simple fraction in lower terms.*

For, let A/B be a fraction such that A is prime to B, and P/Q an equal fraction;

then $\because A/B = P/Q$, [hyp.

$\therefore A \cdot Q = B \cdot P$;

and $\because B$ measures $A \cdot Q$, and is prime to A, [hyp.

$\therefore B$ measures Q. [ths. 4, 9.

So, A measures P.

and P/Q is not in lower terms than A/B. Q. E. D.

COR. 1. *If a fraction be in its lowest terms, so is every integer power of it.*

COR. 2. *A fraction in its lowest terms can be resolved into but one set of factors and divisors,* A^a, $B^b \cdots G^g$, $H^h \cdots$, *wherein* $A, B, \cdots G, H \cdots$ *are different prime numbers, and* $a, b, \cdots g, h, \cdots$ *are integers, some of them negative.*

FACTORS OF THE HIGHEST COMMON MEASURE OF TWO NUMBERS.

THEOR. 13. *The product of all the common prime factors of two or more numbers, each taken with the least exponent it has in any of the numbers, is the highest common measure of the numbers.*

COR. 1. *Every common measure of two or more numbers is a measure of their highest common measure.*

COR. 2. *If each of two or more numbers be multiplied or divided by the same number, their highest common measure is multiplied or divided by this number.*

COR. 3. *The highest common measure of two or more numbers is not changed by multiplying or dividing either number by a number prime to any of the others.*

QUESTIONS.

1. When is it desirable to change a given fraction to higher terms? how can it be done?

2. If a fraction be in its lowest terms, what is true of its numerator and denominator? of any integer powers of them?

3. Is the fraction 3289/3325 in its lowest terms?

4. If the fraction A/B be not in its lowest terms but A, B have the single common factor F, by what single process can $(A/B)^n$ be reduced to its lowest terms?

5. The entire factors of a fraction are factors of which part of it? the divisors are factors of which part?

In how many ways can either of these parts be factored?

What kind of numbers are the factors of a fraction?

What, the reciprocals of the divisors?

Why are some of the integers $a, b, \cdots h$ of theor. 12 cor. 2 negative? why not all of them?

6. Prove theor. 13, by showing that such a product measures each of the numbers, and that no higher number can measure them all.

7. How many of the common factors of several numbers are found in their highest common measure?

8. In how many of the numbers must a factor be found in order to be a factor of their highest common measure?

9. If a common factor be rejected from two or more numbers and the highest common measure of the quotients be found, what has been done to the highest common measure of the given numbers?

How must the highest common measure that has been found be changed to give that of the original numbers?

10. If two fractions, when reduced to their lowest terms, have different denominators, their sum can not be an entire number.

11. If the denominator of a fraction in its lowest terms have other factors than 2 and 5, the fraction can not be exactly expressed as a decimal.

FACTORS OF THE LOWEST COMMON MULTIPLE OF TWO OR MORE NUMBERS.

THEOR. 14. *The product of all the different prime factors of two or more entire numbers, each with the greatest exponent it has in any of the numbers, is their lowest common multiple.*
[df. l. c. msr., th. 4 cr. 5, th. 9 cr. 5.

COR. 1. *Every common multiple of two or more numbers is a multiple of their lowest common multiple.*

COR. 2. *If each of two or more numbers be multiplied or divided by the same number, their lowest common multiple is multiplied or divided by this number.*

COR. 3. *The product of two numbers is the product of their highest common measure and lowest common multiple.*

COMMON MEASURES AND MULTIPLES OF THREE NUMBERS.

THEOR. 15. *The highest common measure of three numbers is the highest common measure of the highest common measure of any two of the numbers and the third number; and so for the lowest common multiple.*

COMMON MEASURES AND MULTIPLES OF TWO FRACTIONS.

THEOR. 16. *The highest common measure of two fractions in their lowest terms is the quotient of the highest common measure of the numerators by the lowest common multiple of the denominators; and their lowest common multiple is the quotient of the lowest common multiple of the numerators by the highest common measure of the denominators.*

For, let A/B, C/D be two fractions in their lowest terms,

and let F/M be a measure of them in its lowest terms;

then $\because A/B : F/M$, $C/D : F/M$, i.e., AM/BF, CM/DF are entire,

$\therefore F$ is a common measure of A, C, and M a common multiple of B, D; [th. 4 cr. 2, th. 9 cr. 2.

and F/M is highest when F is the highest common measure of A, C, and M the lowest common multiple of B, D.

So, let M/F be a common multiple of A/B, C/D;

then ∵ M/F : A/B, M/F : C/D, *i.e.*, BM/AF, DM/CF are entire,

∴ F is a common measure of B, D, and M a common multiple of A, C, [th. 4 cr. 2, th. 9 cr. 2.

and M/F is lowest when F is the highest common measure of B, D, and M the lowest common multiple of A, C.

QUESTIONS.

1. In the proof of theor. 14, show that each of the numbers is a measure of such a product, and that, if any factor were omitted from this product or taken fewer times, some one of the numbers would no longer be a measure of it.

2. In the lowest common multiple of A, B, what factors of A are found, and what factors of B are among them?

What other factors must be added?

3. Let A, B be any two entire numbers, H their highest common measure, L their lowest common multiple, and let $A = aH$, $B = bH$; let A have the factor C^m and B the factor C^n, and $m > n$: how many times is the factor C in H? in a? in L?

If $m < n$, how many times is C in H? in b? in L?

4. A highest common measure found as in theor. 15 contains all the factors common to all the numbers and no others; and a lowest common multiple contains all their factors.

5. In the proof of theor. 16, if the fractions be multiplied by the lowest common multiple of their denominators, what do the fractions become? How is the highest common measure of these products related to that of the given fractions?

If the fractions be in their lowest terms, can the lowest common multiple of the denominators contain any factor in both numerators? Are the multipliers of the numerators prime to each other?

6. Let A/B, C/D be two fractions in their lowest terms, and let $B = b \cdot F$, $D = d \cdot F$: what is F? what is $b \cdot d \cdot F$?

Multiplying both these fractions by $b \cdot d \cdot F$ gives two integers whose lowest common multiple is bd times the lowest common multiple of A, C, and bdF times that of the fractions.

§ 4. THE HIGHEST COMMON MEASURE.

PROB. 4. TO FIND THE HIGHEST COMMON MEASURE OF TWO OR MORE ENTIRE NUMBERS.

(*a*) *The prime factors of all the numbers known: multiply together all the different prime factors, each with the greatest exponent it has in all of the numbers.* [th. 13.

E.g., of $9a^3b^5c$, $3ab^2cd$, $15ab^2c^6-12ab^4$, the common factors are 3, a, b^2; and the highest common measure is $3ab^2$.

So, $\because\ 2x^2y+6xy^2-6x^3-2y^3=2(x-y)(x+y)(y-3x)$
$16ax^2y-12ax^3-4axy^2=4ax(x-y)(y-3x)$
$10y^3-50xy^2+70x^2y-30x^3=10(x-y)^2(y-3x)$,

$\therefore$ the h. c. msr. of these expressions is $2(x-y)(y-3x)$.

(*b*) *The prime factors not known, two entire numbers: divide the higher number by the lower, the divisor by the remainder if any, that divisor by the second remainder, and so on till nothing remains;* [ths. 2, 7.

at pleasure, suppress from any divisor any entire factor that is prime to the dividend corresponding, and introduce into any dividend any entire factor that is prime to the divisor; [th. 13 cr. 3.

at pleasure, suppress from any divisor and the corresponding dividend any common measure of them, but reserve this measure as a factor of the final result. [th. 13 cr. 2.

The product of the last divisor, as above, by the reserved factors if any, is the highest common measure sought. [ths. 2, 7.

E.g., to find the h. c. msr. of x^2+x-12, $x^2-10x+21$:

write

$$\begin{array}{rr} & x^2+x-12)\,x^2-10x+21\,(1 \\ & x^2+\ \ x-12 \\ \cline{2-2} & -11)\,-11x+33 \\ & x-3\,(x+4 \\ x^2-3x & \\ \cline{1-1} 4x-12 & \end{array}$$

or

$$\begin{array}{rr} & 1\ \ 1\ {}^-12)\,1\ {}^-10\ {}^+21 \\ & 1\ \ 1\ {}^-12 \\ \cline{2-2} & {}^-11)\,{}^-11\ \ 33 \\ 1\ {}^-3 & 1\ {}^-3 \\ \cline{1-1} 4\ {}^-12 & \end{array}$$

and $x-3$ is the measure sought.

So, of $4ax^2+4ax-48a$, $4ax^2-40ax+84a$, $4a$ is a common factor, $x-3$ is the h.c.msr. of the remaining factors, and $4a(x-3)$ is the highest common measure.

QUESTIONS.

Find the highest common measure of:

1. $x-1, \quad x^2-1, \quad x^3-1.$ 2. $1-x^2, \quad (1+x)^2, \quad 1+x^3.$
3. $3(x^3-a^2x), \quad 4(x^2+ax), \quad 5(x^4-a^4), \quad 6(x^8-a^8).$
4. $x^2+2x-3, \quad x^3-7x^2+6x.$ 5. $3x^3-24x-9, \quad 2x^3-16x-6.$
6. $1-2x, \quad 1-4x^2, \quad 1-8x^3. \quad 1-16x^4, \quad 1-32x^5.$
7. $x^5+x^4+x^3+x^2+x+1, \quad x^2-x+1.$
8. $4+5x+x^2, \quad 8-2x-x^2, \quad 12+7x+x^2, \quad 20+x-x^2.$
9. $529(x^2+x-6), \quad 782(2x^2+7x+3), \quad 935(2x^2-3x-2).$
10. $x^3+3x^2+4x+12, \quad x^3-5x^2+4x-20.$

Reduce these fractions to their lowest terms:

11. $\dfrac{x^2-6x+5}{7x^2-12x+5}.$ 12. $\dfrac{1+3x-4x^2-12x^3}{8x^3-4x^2-2x+1}.$

13. $\dfrac{1+x^2+25x^4}{1+3x-15x^3-25x^4}$ 14. $\dfrac{3a^3-18a^2+33a-18}{12a^3-84a+72}.$

15. Given $3a^2-10a+3, \quad 6b^2+7b-20, \quad m^3+n^3,$ all prime to each other: find the highest common measure of
$(3a^2-10a+3)^3\cdot(6b^2+7b-20)^2\cdot(m^3+n^3)^4,$
$(3a^2-10a+3)^2\cdot(6b^2+7b-20)^3\cdot(m^3+n^3).$

Find the highest common measure of:

16. $a^4+4a^3+4a^2, \quad a^3b-4ab, \quad a^4b+5a^3b+6a^2b.$
17. $x^2-(y+1)^2, \quad y^2-(x+1)^2, \quad 1-(x+y)^2.$
18. $a^6-b^6, \quad a^2+ab+b^2, \quad a^4+a^2b^2+b^4.$
19. $x^3+4x^2+4x+3, \quad x^3-x^2-x-2.$
20. $4x^4+9x^3+2x^2-2x-4, \quad 3x^3+5x^2-x+2.$
21. $2x^3+(2a-9)x^2-(9a+6)x+27, \quad 2x^2-13x+18.$
22. $4a^2-4ax-15x^2, \quad 6a^2+7ax-3x^2.$
23. $nx^3+3nx^2y-2nxy^2-2ny^3, \quad 4mx^3+mx^2y-2mxy^2-3my^3.$
24. $x^4-px^3+(q-1)x^2+px-q, \quad x^4-qx^3+(p-1)x^2+qx-p.$
25. $x^3+(4a+b)x^2+(3a^2+4ab)x+3a^2b,$
$x^3+(2a-b)x^2-(3a^2+2ab)x+3a^2b.$
26. $a^3e^{2x}+e^{2x}-a^3-1, \quad (a-2+a^{-1})\cdot(e^x-2+e^{-x}).$

NOTE 1. The arrangement of terms may be as to the rising powers of the letter of arrangement, or as to the falling powers.

E.g., $2x^3+11x^2+20x+21$ and x^3-x-6;
or $21+20x+11x^2+2x^3$ and $6+x-x^3$.

That arrangement is commonly best which makes the trial divisor smallest; and at any step of the work the highest or the lowest term of the divisor may be used as trial divisor.

E.g., to find the h. c. msr. of x^3+3x^2+5x+3, x^3+6x^2+9x+4,

write

$$\begin{array}{r|l|r}
x^3+3x^2+5x+3 & 1 & x^3+6x^2+9x+4 \\
 & & \underline{x^3+3x^2+5x+3} \\
9x^2+12x+3 & 3-7x & 3x^2+4x+1 \\
\hline
x^3-6x^2-7x & & \\
\underline{-21x^3-28x^2-7x} & & \\
\underline{22x^3+22x^2 \quad (22x^2} & & \\
x+1 & 3x+1 & 3x^2+3x \\
 & & \overline{x+1}
\end{array}$$

(*c*) *The prime factors not known, three or more entire numbers: find the highest common measure of any two of them* (preferably the lowest), *the highest common measure of this measure and the next number, and so on.*

E.g., to find the highest common measure of

$$x^2+x-12, \quad x^2-10x+21, \quad x^3-6x^2-19x+84:$$

then $\because$ of x^2+x-12, $x^2-10x+21$, the h. c. msr. is $x-3$,
and $x-3$ measures $x^3-6x^2-19x+84$,

$\therefore$ $x-3$ is the measure sought. Q.E.F.

(*d*) *Some or all of the numbers fractions: divide the highest common measure of the entire numbers and the numerators by the lowest common multiple* [pr. 5] *of the denominators.* [th. 16.

E.g., to find the highest common measure of the fractions

$$(x^2+x-12)/(x-5), \quad (x^2-10x+21)/(x+5):$$

then $\because$ the h. c. msr. of the numerators is $x-3$, [above.
and the l. c. mlt. of the denominators is x^2-25, [inspection.

$\therefore$ $(x-3)/(x^2-25)$ is the measure sought.

QUESTIONS.

Reduce to lowest terms by means of the highest common measures of their numerators and denominators:

1. $\dfrac{x^{3a}+x^{2a}+x^a+1}{1-x^a+x^{2a}-x^{3a}}$.   2. $\dfrac{x^4-a^4}{a^3+a^2x-ax^2-x^3}$.

3. $\dfrac{a^6+x^6}{x^{10}+a^{10}}$.   4. $\dfrac{a^6-x^6}{a^{10}-x^{10}}$.   5. $\dfrac{xy^{-1}+2+x^{-1}y}{xy^{-2}+x^{-2}y}$.

6. $\dfrac{x^5+y^{-5}}{x^7+y^{-7}}$.   7. $\dfrac{x^5-y^{-5}}{x^7-y^{-7}}$.   8. $\dfrac{x^{-2}+11x^{-1}+30}{9x^{-3}+53x^{-2}-9x^{-1}-18}$.

Find the highest common measure of:

9. $15x^4-9x^3+47x^2-21x+28$,
$20x^6-12x^5+16x^4-15x^3+14x^2-15x+4$.

10. $x^4-x^3-3x^2+5x-2$, $x^5-2x^4-x^3+5x^2-4x+1$.

11. $3x^2+(4a-2b)x-2ab+a^2$, $x^3+(2a-b)x^2-(2ab-a^2)x-a^2b$.

12. $2x^2+3x+1$, $2x^2+5x+2$, $2x^3+5x^2-4x-3$.

13. $12x^3-2x^2-7x+2$, $18x^3-9x^2-8x+4$, $36x^4-25x^2+4$.

14. $3a^2x+6abx+3b^2x$, $12a^2-12ab+3b^2$, $10a^2+5ab-5b^2$.

15. $x^5y+12x^3y+35xy$, $x^6+3x^4-28x^2$, $x^5z-x^3z-56xz$.

16. Prove that the highest common measure of two or more numbers is the reciprocal of the lowest common multiple of their reciprocals.

Find the highest common measure of:

17. 24, $\dfrac{15}{64}$, $\dfrac{9}{16}$, $\dfrac{45}{128}$.   18. $\dfrac{x^3}{y^2}+\dfrac{4x}{y}-\dfrac{x^2}{y}-4$, $\dfrac{x^3}{2y}-\dfrac{x^2}{y}-\dfrac{x^2}{2}+\dfrac{x}{y}$.

19. $\dfrac{x^2-4x}{x^2+y^2}$, $\dfrac{x^2-8x+16}{x^4-x^2y^2+y^4}$, $\dfrac{x^2-2x-8}{x^6+y^6}$.

20. $\dfrac{6x^2+10x-24}{x^3+5x^2+6x}$, $\dfrac{2x^2-2x-24}{x^3y-x^2y-6xy}$, $\dfrac{8x^2+22x-6}{x^2y-9y}$.

21. $\dfrac{x^2+2cx+c^2}{(x+a)(x+b)}$, $\dfrac{x^2+(c-b)x-bc}{x^2+(a+c)x+ac}$, $\dfrac{3x^2-3c^2}{x^2+(b+c)x+bc}$.

§ 5. THE LOWEST COMMON MULTIPLE.

PROB. 5. TO FIND THE LOWEST COMMON MULTIPLE OF TWO OR MORE NUMBERS.

(*a*) *The prime factors of all the numbers known: multiply together all the different prime factors, each with the greatest exponent it has in any of the numbers.* [th. 14.

E.g., of $9ab^2c^{-3}$, $12a^2b^3d^4$, $15a^4+21a^2bd$, the prime factors, in their highest powers, are 2^2, 3^2, a^2, b^3, c^0, d^4, $5a^2+7bd$,

and the lowest common multiple is $180a^4b^3d^4+252a^2b^4d^5$.

(*b*) *The prime factors not known, two entire numbers: divide either number by their highest common measure and multiply the quotient by the other number.* [th. 14 cr. 3.

E.g., to find the lowest common multiple of

$x^2+x-12,\quad x^2-10x+21$:

then their h. c. msr. is $x-3$, and the multiple sought is

$(x^2+x-12):(x-3)\cdot(x^2-10x+21)=x^3-6x^2-19x+84$.

(*c*) *The prime factors not known, three or more entire numbers: find the lowest common multiple of any two of the numbers* (preferably the highest), *the lowest common multiple of this multiple and the next number, and so on.*

E.g., to find the lowest common multiple of

$x^3+6x^2+9x+4,\quad x^3+2x^2-11x-12,\quad x^3-7x-6$:

then the l. c. mlt. of x^3+6x^2+9x+4, $x^3+2x^2-11x-12$ is

$(x+1)^2(x+4)(x-3)$,

and the l. c. mlt. of $(x+1)^2(x+4)(x-3)$, x^3-7x-6 is

$(x+1)^2(x+4)(x-3)(x+2)$.

(*d*) *Some or all of the numbers fractions: divide the lowest common multiple of the entire numbers and numerators by the highest common measure of the denominators.*

E.g., to find the lowest common multiple of the fractions

$(x^2-y^2)/(a^2-a-20),\quad (x^6-y^4)/(a^2+6a+8)$:

then ∵ the l. c. mlt. of x^3-y^2, x^6-y^4 is x^6-y^4,

and the h. c. msr. of a^2-a-20, a^2+6a+8 is $a+4$,

∴ the multiple sought is $(x^6-y^4)/(a+4)$.

QUESTIONS.

Find the lowest common multiple of:

1. $6x^2-x-1,\quad 2x^2+3x-2.$ 2. $a+1,\quad 1-a,\quad a^2+a+1.$
3. $a^2-b^2,\quad a^2+2ab+b^2.$ 4. $x^3-6x^2-x+30,\quad x^3+3x^2-4.$

Reduce to their lowest common denominators:

5. $\dfrac{x^3+y^3}{x^3-y^3},\quad \dfrac{x^3-y^3}{x^3+y^3},\quad \dfrac{x^2+xy+y^2}{x^2+y^2},\quad \dfrac{x^2-xy+y^2}{x^2-y^2},\quad \dfrac{x-y}{x+y},\quad \dfrac{x+y}{x-y}.$
6. $\dfrac{a+b}{a-b},\quad \dfrac{a-b}{a+b},\quad \dfrac{a^2+b^2}{a^2-b^2},\quad \dfrac{a^2-b^2}{a^2+b^2},\quad \dfrac{a^3+b^3}{a^3-b^3},\quad \dfrac{a^3-b^3}{a^3+b^3}.$

Add the fractions:

7. $\dfrac{1}{2(a+x)},\quad \dfrac{3}{4(a-x)},\quad \dfrac{5}{6(a^2-x^2)}.$ 8. $\dfrac{2}{x},\quad \dfrac{3}{1-2x},\quad \dfrac{2x-3}{4x^2-1}.$
9. $5/6(a^2+x^2),\quad 7/8(a^2-x^2),\quad 9/10(a^2+ax+x^2).$
10. $1/4x^3(x+y),\quad 1/2x^2(x^2+y^2),\quad 1/4x^3(x-y),\quad 1/2x^2(x^2-y^2).$

Find the lowest common multiple of:

11. $x^2-4a^2,\quad x^3+2ax^2+4a^2x+8a^3,\quad x^3-2ax^2+4a^2x-8a^3.$
12. $4(x^2-y^2)(x-y)^3,\quad 12(x^4-y^4)(x-y)^2,\quad 20(x^2-y^2)^3.$
13. $x^2-4a^2,\quad x^3+6ax^2+12a^2x+8a^3,\quad x^3-6ax^2+12a^2x-8a^3.$
14. $m^2n^2-2mn^3+n^4,\quad x^2+2xy+y^2,\quad mnx-nx-ny+mny.$
15. $x^4-y^4,\quad x^3-y^3,\quad x^2-y^2,\quad x^2-2xy+y^2,\quad x^2+2xy+y^2.$
16. $a^2m^2+8am+16,\quad a^2-4,\quad a^2+3a+2,\quad a^2-3a+2.$
17. $x^3-6x^2+11x-6,\quad x^3-9x^2+26x-24,\quad x^3-8x^2+19x-12.$
18. $a^4+a^2b^2+b^4,\quad a^4+3a^3b+4a^2b^2+3ab^3+b^4.$
19. $4a^6-4a^4-29a^2-21,\quad 4a^6+24a^4+41a^2+21.$
20. $\dfrac{x^2+x}{2-2x^2},\quad \dfrac{x^3-x}{8+8x},\quad \dfrac{x^4+x}{12+12x^2},\quad \dfrac{x^2-x}{1+2x+2x^2+x^3}.$
21. $\dfrac{x^2+5x+4}{x^2-4x+3},\quad \dfrac{x^2+2x-8}{x^2-x-12},\quad \dfrac{x^2+7x+12}{x^2+x-20},\quad \dfrac{x^2-3x-4}{x^2+2x-15}.$
22. $\dfrac{x^2-y^2-z^2+2yz}{x^2+2xy+y^2-z^2},\quad \dfrac{x^2-y^2+z^2-2xz}{-3x^2+3y^2-6yz+3z^2}.$
23. $\dfrac{2x^3+x^2-5x-3}{x^4+2x^3+2x^2+x},\quad \dfrac{2x^3+5x^2-x-6}{x^3-2x-1},\quad \dfrac{x^2-1}{x^4-x^2-2x-1}.$

§ 6. QUESTIONS FOR REVIEW.

Define and illustrate:

1. An integer; an entire function of one letter; an entire number.

2. A multiple of an integer; a measure.

3. A multiple of an entire function of a letter; a measure.

4. An entire factor of an entire number; a linear factor.

5. A common multiple of two integers; a common measure.

6. A common multiple of two entire functions of a letter; a common measure.

7. The lowest common multiple of two or more entire numbers; the highest common measure.

8. A prime integer; a prime entire function of one letter.

9. Two numbers prime to each other; a composite number.

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10. State the axioms that relate to the sum, difference, and product of two entire numbers.

11. Show how a common measure of two entire numbers is related to their sum; to their difference; to the sum and the difference of any multiples of them.

12. State Euclid's process for finding the highest common measure of two entire numbers, with proofs and illustrations.

Show that every remainder so found is the difference of two multiples of the given numbers; and that, if the numbers be prime to each other, two multiples of them can be found whose difference is either a unit or some expression that is free from the letter of arrangement.

State and prove the converse of this proposition.

Prove that these statements are true:

13. If an entire number be prime to two or more such numbers, it is prime to their product.

14. If two entire numbers be prime to each other, so are any positive integer powers of them.

15. If an entire number measure the product of two such numbers and be prime to one of them, it measures the other.

16. If an entire prime number measure the product of two or more entire numbers, it measures at least one of them.

17. If an entire prime number measure a positive integer power of an entire number, it measures the number, and if the number, then the power.

18. If an entire number be measured by two such numbers that are prime to each other, it is measured by their product.

19. An entire number can be resolved into prime factors in but one way.

20. A common measure of two or more entire numbers can contain no factor that is not in all of them.

21. If fx, an entire function of x, be divided by $x-a$, the remainder is fa; and if $fa=0$, $x-a$ is a measure of fx.

22. If the numerator and denominator of a fraction be prime to each other, the fraction is in its lowest terms; and so is every integer power of it.

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23. Which factors of two entire numbers are factors of their highest common measure? of their lowest common multiple? What effect has it upon their highest common measure to multiply or divide either of the numbers by a number that is not a factor of the other number? by a number that is a factor of the other? upon the lowest common multiple? to multiply or divide both numbers by the same number?

24. What is the product of the highest common measure and the lowest common multiple of two numbers?

Give the general rule, with reasons and illustrations, for:

25. Factoring an integer; factoring a polynomial of known type-form; factoring an entire function of one letter; finding the linear factors of an entire function of one letter.

26. Finding the highest common measure and the lowest common multiple of two or more entire numbers whose prime factors are known; of two entire numbers whose prime factors are not known; of three or more entire numbers whose prime factors are not known; of fractions.

V. VARIATION, PROPORTION, INEQUALITIES, AND INCOMMENSURABLE NUMBERS.

Hitherto concrete numbers, integers or fractions, have been found by counting like entire units,—apples, horses, guests—or by measuring by some definite unit, or part of a unit, and counting the number of times the unit is contained in the thing measured. Abstract units and simple fractions express repetitions and partitions and their combinations.

Such numbers are *commensurable numbers.*

Hitherto also a number has been thought of as something fixed and definite, and if a letter were used to denote a number, it was some fixed and definite number.

Such numbers are *constants.*

But now come two new notions: that of changing values, *variables;* and that of numbers, definite and distinct, which however cannot be expressed by repetitions and partitions, *incommensurable numbers.*

§ 1. VARIATION.

If a boy count apples—one, two, three $\cdots$, as he drops them into a basket, the number of apples in the basket changes and increases; or, if he have twelve apples at the start and take them out one by one, the number left changes and decreases—twelve, eleven, ten, $\cdots$, and the number of apples in the basket is a variable. Or, if he count the horses that pass through a gate into a field—one, two, three, $\cdots$, or the guests as they rise from table, or the 3's he gets when he reduces the simple fraction 1/3 to a decimal: in all these cases the numbers so found are variables.

So, if the cross-section of the trunk of a growing tree be a circle, it is a circle whose radius, circumference, and area are all variables; and a growing peach is a sphere whose radius, circumference, surface, and volume are variables.

So, with a sum of money at interest, the principal and rate are constants; the time and accrued interest are variables.

QUESTIONS.

1. Show that a variable may increase or decrease by regular additions or subtractions, or by irregular ones.

2. In annexing successive 3's to the decimal expression of the fraction 1/3, are the successive additions to the value of the variable the same or different?

3. Is the number 1/9 a variable? is its decimal expression a variable? can the same number, then, be both a variable and a constant, or is 1/9 not equal to its decimal expression?

4. Is the reciprocal of a variable a constant or a variable? How does the reciprocal of an increasing variable change? the opposite? the opposite of the reciprocal?

5. In general, is the sum of two variables a constant or a variable? their difference? their product? their quotient?
Show by examples that the sum, the difference, the product, and the quotient, of two variables may be constants.

6. In the case of an express train running over a long, level, straight track, with the same pressure of steam, which of the following elements are variables: the time since the train started, the distance it has run, the speed, the relation of the distance to the speed, its relation to the time?

7. If the wine in a full cask run into an empty one, what two variables are there during the process? what constant?
Do these two variables vary in the same way?

8. The diagonal of a square whose side is a is $\sqrt{2a^2}$; what is the diagonal of a square whose side is b?
What relation does the diagonal of a square bear to a side?
If the length of a side change, does the diagonal change?
Does the relation between the side and diagonal change?

9. What is the area of a square whose side is a? of one whose side is b? What is the relation of the area to the side?
Is the area of a growing square a fixed number of times the side? What definite law connects these two magnitudes?
If the length of a side can not be exactly expressed, what effect has that on this law?

CONTINUOUS AND DISCONTINUOUS VARIABLES.

If a concrete variable, in passing from one value to another, pass through every intermediate value, it is a *continuous variable;* otherwise it is a *discontinuous variable.*

So, the abstract ratios of these concrete variables to the constant measuring unit are continuous or discontinuous variables.

E.g., if a man has waited two hours for an incoming steamer, he has also waited an hour, an hour and a quarter, an hour and a half, and every other portion of time less than two hours that can be named or conceived of; and he has been conscious of the continuous passage of time.

So, if he run along the street he knows that he cannot get from one fixed point to another without going through every intermediate point of some path, and that the distance run has a continuous growth.

But, of the regular polygons inscribed in a circle, one may have three sides, another four, another five, and so on; but no one can have four and a half sides, nor five and a quarter sides. The number of sides, the perimeter, and the area are all discontinuous variables.

RELATED VARIABLES.

If two variables be so related that the value of one of them depends upon that of the other, the first variable is a *function* of the other. [compare IV, § 2.

E.g., with a given principal and rate, the accrued interest is a function of the time.

So, the circumference and area of a growing circle are functions of the radius.

So, of the regular polygons inscribed in a given circle, the perimeter and area are functions of the number of sides.

A variable may be a function of two or more variables.

E.g., the volume of a stick of timber is a function of its length, breadth and thickness; and the cost is a function of the length, breadth, thickness, and cost per foot.

QUESTIONS.

1. Can a variable be continuous, and its reciprocal be discontinuous? its opposite? its square? its square root?

2. State whether the numbers below are constants or variables, and if variables, whether continuous or discontinuous:

the length of a line revolving about the centre of a circle as a pivot and reaching to the circumference;

such a line revolving about any other point than the centre;

the principal at simple interest; at compound interest;

the size of an angle if the bounding lines be lengthened;

the number of telegraph poles passed by the electric current.

3. If a falling body has at one moment a velocity of 30 ft. a second, and later a velocity of 40 ft.; how many other different velocities has it had between these times?

4. What relation does the perimeter of a square bear to the length of one side? If the side, for any reason, can not be measured, does that fact affect this relation?

5. The circumference of a circle is readily seen to be somewhat more than three times the diameter; the exact relation is found by geometry, and it is usually represented by the Greek letter π, read *pie*, whose value is nearly 3.1416: what is the relation of the circumference of a circle to its radius? does this relation depend in any way on the length of the radius? on the length of the circumference?

6. If the radius of a circle increase, does the circumference increase at the same rate or at a greater or less rate? the area?

7. As interest is computed in business, for days and not for fractions of a day, thus making the time discontinuous, is the accrued interest a continuous or a discontinuous function of the time? the amount?

8. If x be a continuous variable, is x^3-2x^2+x+5 a constant, or a continuous, or a discontinuous, function of x? $1/x$?

9. If the diameter of a peach grow continuously, does the circumference grow continuously? the surface? the volume? the weight? the value?

RELATIVE VARIATION.

One number *varies directly* as another if when the second is doubled so is the first, when the second is tripled so is the first, when the second is halved, so is the first, and so on.

E.g., with a constant principal and rate the interest accrued varies directly as the time.

So, with men of equal efficiency, the amount of work done in a day varies directly as the number of men employed.

The sign of variation is $\propto$, read *varies as.*

E.g., if i stand for simple interest and t for time, then $i \propto t$.

So, if w stand for the work done in a day and m for the number of men, then $w \propto m$.

If two abstract numbers, or two concrete numbers of the same kind, vary in such wise that some relation between their magnitudes shall continue to hold true, this relation may be expressed by an equation.

E.g., if d be the diameter of a circle and r its radius;

then $d \propto r$, and $d=2r$.

One number *varies inversely* as another, if when the second is doubled the first is halved, when the second is tripled the first is trisected, when the second is halved, the first is doubled, and so on. If the numbers be abstract one varies directly as the reciprocal of the other.

E.g., the time of running a fixed distance varies inversely as the speed.

So, the illuminating power of a light varies inversely as the square of its distance.

So, if x, y, be two abstract numbers and x vary inversely as y;

then $x \propto 1/y$, and $x=k/y$, wherein k is some constant.

One number *varies jointly* as two others if it vary directly as their product.

E.g., if w stand for the wages earned, m for the number of men and t for the time, then $w \propto m \cdot t$, or $w=k \cdot m \cdot t$.
[k, a constant.

A number may vary directly as one number and inversely as another.

E.g., the time of running varies directly as the distance run and inversely as the speed.

So, the value of a fraction varies directly as the numerator and inversely as the denominator.

QUESTIONS.

1. What is meant by saying that the circumference of a circle varies as the diameter? that the radius varies as the circumference? that the area varies as the square of the radius?

If a stone be dropped into a pond of water, how many of these relations will be true at each and every stage of the growing circular wave?

2. Does the light received from a lamp vary as its distance? how then? the entire cost as the number of like articles bought? the time needed for a piece of work as the number of men?

3. If a lady spend the same sum for gloves every year, but the price increase, how does the number of pairs bought vary?

4. Of rectangles having the same area, how does the width vary? the length? the perimeter? the diagonal?

5. If the sum of two numbers be constant, will one vary as the other? inversely as the other? if the product be constant? the quotient? the difference?

6. If at different times varying numbers of like things be bought, and at varying prices, how do the bills vary?

7. If the dividend be constant and the divisor vary, how does the quotient vary? if the divisor be constant and the dividend vary? if the dividend and divisor both vary?

May the dividend and divisor both vary and the quotient remain constant?

8. In simple interest is the principal a constant or a variable? with a fixed principal and rate, how do the interest and time vary? that a fixed principal, at interest for various periods of time, may bring in the same interest for each period, how must the rate be related to the time?

§ 2. PROPORTION.

The *ratio* of one quantity to another quantity of the same kind is the number found by measuring the first quantity by the other as a unit; *i.e.*, it is the multiplier that, acting on the second quantity as a unit, gives the first quantity as result.

The ratio of two like numbers is the quotient of the first by the other, and it is an abstract number.

E.g., the ratio of 12 bushels of wheat to 4 bushels of wheat is 3, and the ratio of 4 bushels to 12 bushels is 1/3.

The ratio of a to b is written $a:b$ or a/b.

No ratio is possible between unlike numbers.

The ratio $a:b$ is the *direct ratio* of a to b, and $b:a$ is the *inverse ratio*.

In a direct ratio, the dividend is the *antecedent* and the divisor is the *consequent*.

The equation of two ratios is a *proportion*, and the four numbers forming the two ratios are the *terms* of the proportion, or the four *proportionals*.

E.g., if the ratio of a to b equals that of c to d, this fact may be stated in a proportion in three ways:

$a:b::c:d, \quad a/b=c/d, \quad a:b=c:d.$

The first form is read, *a is to b as c is to d;* and the others more briefly, *a to b equals c to d.*

A proportion between concrete numbers is but an expanded statement of direct variation.

E.g., 2 years : 3 years = 2 years' interest : 3 years' interest means that, the principal and rate being constant, the interest varies as the time.

The first and last terms of a proportion are the *extremes*, the second and third the *means*, the first and third the *antecedents*, the second and fourth the *consequents*.

If the means of a proportion be the same number, the common term is the *mean proportional*, and the last term is the *third proportional*.

E.g., in the proportion $a:b=b:c$, b is the mean proportional between a, c, and c the third proportional to a, b.

If three or more ratios be equal, they may be written in succession, with the sign = between them. Such an expression is a *continued proportion*.

E.g., $a:b=c:d=e:f=g:h$.

QUESTIONS.

1. Can a ratio be concrete? can it be negative?

2. If two numbers be equal, what is their ratio? opposites? if the antecedent be the larger? the consequent?

3. What two relations between the antecedent and consequent make the direct and inverse ratios equal?

4. If a ratio be zero, which term is zero? if infinite?

5. Name two other numbers that have to each other the same ratio as 10 has to 5, and write these four numbers as a proportion. Find other pairs of numbers having the same ratio, and of them all make a continued proportion.

6. As a statement of variation, interpret the proportion
2 hrs. : 5 hrs. = distance run in 2 hrs. : distance run in 5 hrs.

7. As a proportion write the statement that the areas of circles vary as the squares of their radii.

8. If the antecedents of a proportion be equal, what is true of the consequents? if one antecedent be ten-fold the other?

9. If the first antecedent be the square of its consequent, is the second antecedent likewise the square of its consequent?

10. Of the continued proportion at the top of the page, make six different simple proportions.

11. What effect is produced on a ratio by doubling the antecedent? the consequent? both terms?

12. Regarding a proportion as an equation between two fractions, show why it is allowable to multiply or divide both antecedents by the same number? both consequents? both terms of one ratio? all four terms? to multiply one antecedent and divide the other consequent by the same number?

PROPERTIES OF PROPORTIONS.

A proportion is but an equation whose members are fractions; and the principles already established for fractions and for equations apply directly to the proportions used in the proof of the theorems below.

In these proofs, the proportionals are abstract numbers.

THEOR. 1. *In any proportion, the product of the extremes equals that of the means.*

Let $a:b=c:d$, then will $ad=bc$.

For $\because a/b=c/d$, [hyp.

$\therefore ad=bc$. Q.E.D. [mult. by bd; ax. mult.

COR. 1. *Either extreme is the quotient of the product of the means when divided by the other extreme; and so for the means.*

COR. 2. *A mean proportional is the square root of the product of the extremes.*

THEOR. 2. *If the product of two numbers equal the product of two others, the four numbers may form a proportion, in which the factors of one product shall be the extremes and those of the other product the means.*

Let $ad=bc$; then $a:b=c:d$, $a:c=b:d$.

For the first proportion, divide both members of the equation $ad=bc$ by bd; for the other, divide by cd.

THEOR. 3. *In any proportion, the two means may change places.* [alternation.

Let $a:b=c:d$; then $a:c=b:d$.

Prove by aid of theors. 1, 2.

THEOR. 4. *In any proportion, the consequents may change places with their antecedents.* [inversion.

Let $a:b=c:d$; then $b:a=d:c$.

Prove by aid of theors. 1, 2.

THEOR. 5. *In any proportion the sum of the first two terms is to the first term or the second as the sum of the last two terms is to the third term or the fourth.* [composition.

QUESTIONS.

1. What is the test of the correctness of a given proportion?

2. Supply the missing terms in the proportions below:

$6:4=9:\ \ ;\qquad 7:\ \ =-35:-5;\qquad -9:\ \ =\ \ :16;$

$\ \ :3=0:10;\qquad \ \ :16=4:\ \ ;\qquad \ \ :-9=16:\ \ .$

Is there more than one way of completing the third of these proportions? how many ways? could the two means be equal?

3. How is theor. 2 related to theor. 1?

4. In proving theor. 3, what multiplier will change the given ratio a/b to the desired one, a/c?

5. How is the reciprocal of a number found? Prove theor. 4.

6. Clear the equation $a/b=c/d$ of fractions, and see what multiplier will make the first member d/c; and so make a new proof of theor. 4.

7. If the extremes of a proportion be to each other as the means, the ratios are each unity.

8. If two terms of one proportion be the same as the two like terms of another, the four terms that are left may form a proportion.

9. The direct ratio of two numbers is the inverse ratio of their reciprocals.

10. May the reciprocals of any four proportionals form a proportion? their opposites? the opposites of their reciprocals?

11. May any four proportionals be written in reverse order? their opposites? the opposites of their reciprocals?

12. If $m:a=n:b$ and $c:m=d:n$, then $a:b=c:d$.
So, if $a:m=b:n$ and $c:d=m:n$.

13. If $a:m=n:b$ and $c:m=n:d$, then $a:1/b=c:1/d$
So, if $a:m=n:b$ and $m:c=d:n$.

14. No single number can be added to each of the proportionals a, b, c, d and leave the sums in proportion.

15. If there be a single number, not 0, that may be added to each antecedent without destroying the proportion, what relation have the terms of the proportion?

THEOR. 6. *In any proportion the difference of the first two terms is to the first term or the second as the difference of the last two terms is to the third term or the fourth.* [division.

COR. 1. *The sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

COR. 2. *The sum of the antecedents is to their difference as the sum of the consequents is to their difference.*

THEOR. 7. *If two proportions be multiplied together, or if one be divided by the other, term by term, the results are proportional.*

THEOR. 8. *Like powers and like roots of the terms of a proportion are proportional.*

THEOR. 9. *In a continued proportion, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

For, let $a:b=c:d=e:f=\cdots$

then $\because a/a=b/b, \quad c/a=d/b, \quad e/a=f/b, \cdots$

$\therefore (a+c+e+\cdots):a=(b+d+f+\cdots):b,$ [add.

$\therefore (a+c+e+\cdots):(b+d+f+\cdots)=a:b.$ [alternation.

Of these nine theorems, only theors. 4, 5 are always directly applicable to concrete numbers; if, however, all the terms be of the same kind, theors. 3, 9 also apply.

Moreover, though the terms of a proportion be all concrete, their ratios are abstract, and so are the products, quotients, powers and roots of these ratios; and these results may be used as operators on concrete units.

E.g., if a days : b days $=$ \$$c$: \$$d$, and e men : f men $=$ \$$g$: \$$h$,

then ae days' labor : bf days' labor $=$ \$$cg$: \$$dh$.

For the proportions give the abstract equations

$$a/b=c/d, \quad e/f=g/h, \quad ae/bf=cg/dh,$$

and $\because$ ae days' labor : bf days' labor $= ae:bf$,

and \$$cg$: \$$dh = cg:dh$,

$\therefore$ ae days' labor : bf days' labor $=$ \$$cg$: \$$dh$.

QUESTIONS.

1. To each side of the equation $a/b=c/d$ add 1, and reduce these two mixed numbers to fractions; from the resulting equation find a hint for the proof of theor. 6.

Write the proportion by inversion, then prove theor. 5.

2. Divide the equation $(a+b)/a=(c+d)/c$ by the equation $(a-b)/a=(c-d)/c$, member by member, and prove theor. 6 cor. 1.

So, from the equation $a/b=c/d$ get $a/c=b/d$, and prove theor. 6 cor. 2.

3. One fraction can be divided by another by dividing the numerator and denominator of the first by the like terms of the other; and such a process is equivalent to the usual one of multiplying by the divisor inverted; why is the latter rule oftener used than the other?

Apply this method in proving theor. 7.

4. If the first two terms of a proportion be squared, by what is the first side of the equation multiplied? if the other two be also squared, is the same multiplier used or a different one? Hence prove theor. 8.

5. Why may the two antecedents of a proportion be multiplied by one number and the two consequents by another?

6. Show why theors. 1, 7, 8 are of no direct use when all the terms are concrete numbers; and why theor. 2, but not its converse, may apply to concrete numbers.

7. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, then a, b, c, d are proportionals.

8. If $a:b=c:d=e:f$, and h, k, l be any numbers, then

$$a:b=(ha+kc+le):(hb+kd+lf),$$

and $$a^n:b^n=(ha^n+kc^n+le^n):(hb^n+kd^n+lf^n).$$

9. The distance fallen varies as the square of the time; a body falls 16 feet in one second: how far does it fall in two seconds? in three seconds? in the third second? in five seconds? in the last two of the five seconds? How high is a tower from whose top a stone falls in $3\frac{3}{4}$ seconds?

§ 3. INEQUALITIES.

LARGER-SMALLER INEQUALITIES.

One concrete number is *larger* than another of the same kind if it contain more units than the other; and *smaller* if it contain fewer units. The positive or negative quality of the numbers is not thought of, but only their magnitudes.

One abstract number is larger than another if it give a larger result when acting on the same unit.

E.g., if A have \$50 and owe \$30, and B have \$60 and owe \$80;

then A's assets are smaller than B's, and so are his debts,
A's assets are larger than his debts, and B's are smaller,
A's net assets are as large as B's net debts:

and $^+50 \lessdot {}^+60$; $^-30 \lessdot {}^-80$; $^+50 \gtrdot {}^-30$; $^+60 \lessdot {}^-80$.

AXIOMS.

1. *If of three numbers the first be larger than the second, and the second be equal to or larger than the third, then is the first number larger than the third.*

2. *If one number be larger than another, and if each of them be multiplied by the same number or by equal numbers, then is the first product larger than the other.*

3. *If one number be larger than another, and if each of them be divided by the same number or by equal numbers, then is the first quotient larger than the other.*

4. *If one number be larger than another, and if the same number or equal numbers be divided by each of them, then is the first quotient smaller than the other.*

5. *If one set of numbers be larger than another set of as many more, each than each, then is the product of the first set larger than the product of the others.*

6. *If one number be larger than another, and if like positive powers or roots of them be taken, then is the power or root of the first larger than that of the other.*

7. *If one number be larger than another, and if like negative powers or roots of them be taken, then is the power or root of the first smaller than that of the other.*

Not all of these axioms are axioms in the sense of truths too elementary to admit of proof, for some are directly derivable from others, but they are all self-evident.

QUESTIONS.

1. When a number is made larger, what effect is produced on its reciprocal? its opposite? the opposite of its reciprocal?

2. Which is the larger, zero or a negative number?

3. Explain the axioms, and illustrate each of them by an inequality between known numbers.

Show which of them are deducible from the others.

4. Prove the following statements, with reference to larger-smaller inequalities, giving axioms as authority:

both members of an inequality may be multiplied or divided by the same number, or raised to the same positive power;

the products of the corresponding members of several inequalities may be taken without changing the sign of inequality;

but if the same operations be performed on the reciprocals of both members, the sign of inequality must be reversed.

5. Show by trial that adding the same number to both members of a larger-smaller inequality will sometimes reverse the sign of inequality.

So, subtracting the same number from both members.

So, dividing two such inequalities member by member.

6. Of what numbers are negative powers larger than the like positive powers? of what numbers are positive powers smallest when the exponents are largest?

7. Show that if both terms of a proper fraction be positive and to both the same positive number be added, the fraction is made larger thereby.

So, that an improper fraction is thus made smaller.

Which number is the smallest:

8. $\sqrt[2]{2}$, $\sqrt[4]{4}$, $\sqrt[8]{8}$? 9. $\sqrt[3]{3}$, $\sqrt[6]{6}$, $\sqrt[9]{9}$? 10. $\sqrt[5]{5}$, $\sqrt[10]{10}$, $\sqrt[15]{15}$?

11. $(1/2)^{1/2}$, $(1/4)^{1/4}$, $(1/8)^{1/8}$? 12. $(1/3)^{1/3}$, $(1/6)^{1/6}$, $(1/9)^{1/9}$?

13. $(1/5)^{1/5}$, $(1/10)^{1/10}$, $(1/15)^{1/15}$? 14. $3\cdot5\cdot7\cdot9$, $4^2\cdot8^2$, 6^4?

GREATER-LESS INEQUALITIES.

One number is *greater* than another number of the same kind if the remainder be positive, when from the first the other is subtracted, and *less* if the remainder be negative.

Of positive numbers the larger is also the greater, but of negative numbers the smaller is the greater; and any positive number, however small, is greater than any negative number of the same kind, however large.

The signs are $>$, *greater than*; $<$, *less than*.

E.g., $^{+}50 < {}^{+}60$, $^{-}30 > {}^{-}80$, $^{+}50 > {}^{-}30$, $^{+}60 > {}^{-}80$.

AXIOMS.

8. *If of three numbers the first be greater than the second, and the second be equal to or greater than the third, then is the first number greater than the third.*

9. *If one number be greater than another, and if the same number or equal numbers be added to them, then is the first sum greater than the other.*

10. *If one number be greater than another, and if the same number or equal numbers be subtracted from them, then is the first remainder greater than the other.*

11. *If one number be greater than another, and if they be subtracted from the same number or from equal numbers, then is the first remainder less than the other.*

12. *If one set of numbers be greater than another set of as many more, each than each, then is the sum of the first set greater than the sum of the others.*

13. *If one number be greater than another, and if they be multiplied or divided by the same or equal positive numbers, then is the first product or quotient greater than the other.*

14. *If one number be greater than another, and if they be multiplied or divided by the same or equal negative numbers, then is the first product or quotient less than the other.*

NOTE.—The pupil may compare these axioms with those on page 148, taken in order: he will find that adding and subtracting in greater-less inequalities are analogous to multiply-

ing and dividing in larger-smaller inequalities, and that multiplying and dividing in the one are analogous to finding powers and roots in the other. The note at the top of page 149 applies also to this set of axioms.

QUESTIONS.

1. Can a positive number be less than a negative number? can it be smaller? larger? greater?

2. Name two numbers equally large, but unequal.

3. Which of the pair $-(a+b)$, $(a+b)$ is the larger if a, b be both positive? which is the greater? if a, b be both negative? if a be positive, b negative, and a larger than b?

4. In the pair x, x^2 which is the larger if $x>1$? if x be a positive proper fraction? if x be a negative fraction?

What two values of x make x^2 equal to x?

5. What axiom proves that in a greater-less inequality a term may be transposed from one member to the other by changing its sign? that if the signs of every term be changed the sign of inequality must be reversed? that two like inequalities may be added without changing the sign?

6. Why do not axioms 6, 7 apply to inequalities of this kind?

7. If $a>b$ and $a'>b'$, the elements may have such relations that $a-a'>b-b'$, or $a-a'=b-b'$ or $a-a'<b-b'$.

8. If $2x-7>29$, $3x-5<2x+16$, then $18<x<21$.

9. If 16 more than three times the number of sheep exceeds twice their number and 27, and four thirteenths of their number less one be less than 3, how many sheep are there?

10. Twice a number less 3 is less than the number plus 5; and 11 plus twice the number is less than 3 times the number plus 5: what is the number? is the number definite?

11. If x be any positive number, $x+1/x \not< 2$.

12. If $x+y=s$, $4xy=s^2-(x-y)^2$, and xy is greatest when $x-y=0$; hence show that the product of two numbers whose sum is constant is greatest when the numbers are equal.

13. Show which of axs. 8–14 are derivable from others.

§ 4. INCOMMENSURABLE NUMBERS.

Numbers that arise from the effort to measure any quantity by a unit that has no common measure with it are *incommensurable numbers.*

E.g., a side and a diagonal of the same square are both definite lines, but the length of the diagonal cannot be expressed exactly by any simple fraction in terms of the side, nor that of the side in terms of the diagonal;

i.e., neither line can be got from the other by partition and repetition.

So, the circumference of a circle is incommensurable with its diameter.

So, a period of time expressed exactly either in days, in lunar months, or in solar years, can be expressed exactly in neither of the other units.

An abstract number is incommensurable if it can be written approximately, but not exactly, as a fraction. Such numbers arise from attempts to find operators that, used in a specified way, shall yield certain given results.

E.g., to find an operator that, used twice in succession as multiplier, will double a unit.

1. *No integer can be such operator.*

For the operator $^{\pm}1$ used twice as multiplier gives the unit as result; $^{\pm}2$ gives four times the unit; and other integer operators give still larger results.

2. *No simple fraction can be such operator.*

For, let n/d be any simple fraction in its lowest terms;

then the concrete product unit $\times n/d \times n/d$ is unit $\times n^2/d^2$,

and $\because$ the fraction n^2/d^2 is irreducible, [IV, th. 12 cr. 1.

$\therefore n^2/d^2 \neq 2$.

3. *But a simple fraction can be found that shall very closely approach the operator sought.*

For $\because$ unit $\times 1.4^2 =$ unit $\times 1.96$ and unit $\times 1.5^2 =$ unit $\times 2.25$,

$\therefore$ the operator sought lies between 1.4 and 1.5. [ax. 2.

So $\because$ unit $\times 1.41^2 =$ unit $\times 1.9881$,
and unit $\times 1.42^2 =$ unit $\times 2.0164$,

$\therefore$ the operator sought lies between 1.41 and 1.42.

So it lies between 1.414 and 1.415, between 1.4142 and 1.4143, and so on.

But such operator exists, and is a definite number, which is perfectly expressed in the language of geometry by saying that it is the ratio of the diagonal to the side of the same square, and in the language of algebra by $\sqrt{2}$.

QUESTIONS.

1. If two squares have a side of the same length, how do the two perimeters compare? the two areas? the two diagonals?

If one square be placed on the other, how will the diagonals lie? If the sides of the two squares increase continuously at the same rate, how do the two diagonals increase?

2. If a side of a square be one inch, can the lengths of the diagonals be expressed in inches?

If the side increase continuously to two inches, has its diagonal been at any time measurable in inches? was the side measurable in inches at that time?

But, through all changes of value, and from commensurability to incommensurability, what relation holds true between the two diagonals? between a side and a diagonal?

3. Why could not the decimal expression for $\sqrt{2}$ end with the figure 1? 2? 3? $\cdots$ 9? with what figure must it end so that the square of it may be 2.0?

Show that no decimal fraction ends in 0.

What is thus proved as to $\sqrt{2}$?

4. Is the fraction 2/3 a commensurable number? is the corresponding decimal .666 $\cdots$ a commensurable number?

Have these two expressions the same or different values?

Is one of them more definite than the other?

5. Are the approximate decimal expressions for incommensurable numbers variables or constants? If variables, do they grow continuously or discontinuously?

MULTIPLICATION AND DIVISION.

The product of a concrete number by an abstract number is a number that bears the same relation to the multiplicand that the multiplier bears to unity.

This definition covers the earlier and simpler ones.

E.g., if the multiplier be 2, the product is the double of the multiplicand; if $^-2$, it is the opposite of the double.

So, if the multiplier be 3/4, the product is the triple of a fourth part of the multiplicand.

It also defines multiplication by an incommensurable multiplier; and later it defines multiplication by an imaginary.

E.g., the product six sq. ft. $\times \sqrt{2}$ is a number that has the same relation to six sq. ft. that the length of the diagonal of a square has to that of one of its sides.

So, the product 6 hours $\times$ π is a period of time that bears the same relation to six hours as the length of the circumference of a circle bears to that of its diameter.

So, if simple interest be of continuous growth, the interest varies as the time and $i = p \cdot r \cdot t$, wherein the product $p \cdot r$ is the interest for one year, and the product $p \cdot r \cdot t$ is the interest accrued at any given time, t, whether t stand for a length of time that is commensurable with the unit year or not.

The definition of the product of two abstract numbers is identical with that given on page 6; and the definitions of division and of a quotient are identical with those given on page 16.

MULTIPLICATION ASSOCIATIVE.

THEOR. 10. *The product of three or more abstract numbers* (commensurable or incommensurable) *is the same number, however the factors be grouped.* [I, th. 1.

Let $a, b, c \cdots$ be three or more abstract numbers;
then is their product the same number, however the numbers be grouped.

For, to multiply the concrete product unit $\times a$ by the abstract product $b \times c$ is to multiply the concrete product unit $\times a$ by the abstract number b, and the consequent concrete product unit $\times a \times b$ by the abstract number c, [df. prod. ab. nos.

and so for other factors, and for other groupings of them.

Q. E. D.

QUESTIONS.

1. Draw a square; then another square whose side is a diagonal of the first square: how does the diagonal of the second square compare with the side of the first?

What operator, acting as multiplier on the side of the first square and again on that result, has doubled the given side? Is that operator a commensurable number?

Show that the definitions of multiplication and of a multiplier apply here.

2. Of what two numbers is the diagonal of a square the product? Can the product, or the quotient, of two incommensurable numbers be a commensurable number?

3. How long is the diagonal of a square whose side is one inch? of a square whose side is a inches? What number bears the same relation to unity as the diagonal of a square bears to its side?

4. What two lines are proportional to π and unity?

Is π a commensurable number?

What perfectly definite meaning has it?

What line is exactly expressed by 4π? 10π?

If the circumference of a circle be 10, what is $10/\pi$? $5/\pi$?

If the radius of a circle be 4, what is its area, the area of a circle being half the product of its radius and circumference?

5. If a side of a square be two feet, and if a circle be described upon its diagonal as diameter, what is the length of the circumference of the circle? what is its area?

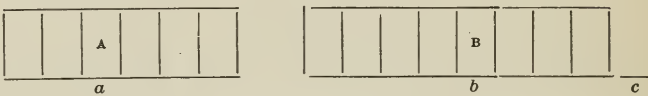
6. By what shall a unit be multiplied three times in succession to quadruple it? four times, to triple it?

MULTIPLICATION COMMUTATIVE.

LEMMA. *Two rectangles that have the same altitude are to each other as their bases.*

(*a*) *The bases commensurable.*

Let A, B be two rectangles of the same height, whose bases, a, b, have a common measure, c;



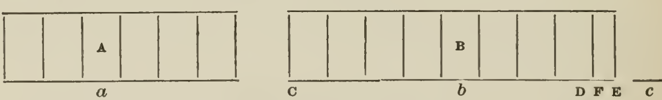
let c be contained in a m times and in b n times; and at the points of division draw perpendiculars to a, b, thus dividing the rectangle A into m parts and B into n parts that may be shown to be all equal by placing one upon another;

then $\because$ A : B $= m:n$ and $a:b=m:n$, [df. ratio.

$\therefore$ A : B $= a:b$. Q.E.D. [ax. equal.

(*b*) *The bases incommensurable.*

Let A, B be two rectangles of the same height, whose bases, a, b, have no common measure;



then if b be not the fourth proportional to A, B, a, that proportional is some line shorter or longer than b.

If possible, let that proportional be CD, shorter than b by the part DE,

and find a measure, c, of a, that is shorter than DE.

Apply c repeatedly to b; then at least one point of division, F, falls between D and E;

draw perpendiculars to a, b as in (*a*);

then $\because$ a, CF are commensurable, [constr.

$\therefore$ A : rectangle on CF $= a$: CF. (*a*)

But A : B $= a$: CD; [hyp.

and $\because$ A : rectangle on CF $>$ A : B and a : CF $<$ a : CD, [ax. 4.

$\therefore$ these two proportions are not both true,

i.e., the hypothesis that A : B $=$ a : some line shorter than b leads to a false conclusion, and is itself false.

So, as may be proved in like manner, the hypothesis that A : B $=$ a : some line longer than b is false,

$\therefore$ it is only left that A : B $=$ $a:b$. Q.E.D.

QUESTIONS.

1. What is a lemma? [consult a dictionary.

2. In case (a), let C be one of the small rectangles into which A, B are divided; express A, B in terms of C; and show why A : B $= m:n$, and why $a:b=m:n$.

3. If a be an incommensurable line, can c be an exact measure of a? and if c be a measure of a, how can a be called incommensurable?

4. In case (b) why must F fall between D and E? Give a reason for the two inequalities stated. How do these inequalities prove that both proportions can not be true?

How is it known which is the false proportion?

5. Such a proof as that of this lemma is called an *indirect proof*, because, instead of proving directly what we wish to establish, we prove that every other possible supposition leads to a false conclusion, and is therefore itself false: why is this proof also called a *proof by exclusion?*

6. Construct two rectangles whose areas are proportional to a side and a diagonal of a square; two, whose areas are proportional to the diameter and circumference of a circle.

7. By aid of the well-known theorem of geometry, "the square of the hypothenuse of a right triangle is the sum of the squares of its legs," construct a line whose length is

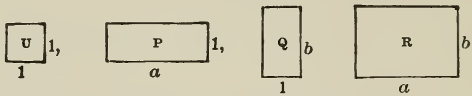
$\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$.

8. Construct a square; a square on the diagonal of this square; a third square on the diagonal of the second square; and a fourth square on the diagonal of the third: what relation have the sides of these squares? what, their areas?

THEOR. 11. *The product of two or more abstract numbers* (commensurable or incommensurable) *is the same number, in whatever order the factors be multiplied.*

(*a*) *two factors a, b.*

For, let U be a square whose side is of unit length, P a rectangle of height 1 and breadth a, Q a rectangle of breadth 1 and height b, R a rectangle of breadth a and height b;



then $\because$ $P = U \times a$, $Q = U \times b$, $R = P \times b = Q \times a$, [lem.
wherein a is the ratio of line a to the unit line, and so of b,

$\therefore$ $R = U \times a \times b = U \times b \times a$,

$\therefore$ the abstract products $a \times b$, $b \times a$, do the same work and are equal.

(*b*) *three or more factors a, b, c.*

The proof is identical with that of I theor. 2 (*d*).

AXIOMS AND THEOREMS THAT APPLY TO INCOMMENSURABLE NUMBERS.

The quality of numbers as denoted by the positive and negative signs applies alike to commensurable numbers and to incommensurable numbers.

The axioms of equality and of inequality, and the definitions and principles of division and reciprocals, are the same for incommensurable numbers as for commensurable numbers; and I theors. 3, 4, relating to reciprocals and division, apply to incommensurable numbers without change in their statement or proof.

So, theor. 5, that addition is commutative and associative, has the same statement, and the proof is as follows:

For, let $a, b, c \cdots$ be any abstract numbers, commensurable or incommensurable,

let these numbers act as operators on any unit,

and let the results be grouped and added in any way;

then ∵ the quantities so found are of the same kind,

and their aggregate is the same, in whatever order they be arranged and however they be grouped,

∴ the several sums of these operators do the same work and are all equal. Q.E.D.

So, theor. 7, that multiplication is distributive as to addition; theors. 8, 9, relating to opposites and subtraction; and theors. 10, 11, 12, relating to integer powers, apply to incommensurable numbers without change.

QUESTIONS.

1. Show that the proof of theor. 11 could not have been complete without the aid of the preceding lemma.

2. State the proof of theor. 11 for three or more factors.

3. What modification is made in I theor. 5, to make it applicable to incommensurable numbers?

4. Review all the theorems that apply without change to incommensurable numbers and give the proofs, remembering that they include such numbers.

§ 5. QUESTIONS FOR REVIEW.

Define and illustrate:

1. A constant; a variable; related variables; a function of a variable; continuous variables; discontinuous variables.

2. Variation; direct variation; inverse variation; joint variation.

3. The ratio of one quantity to another quantity of the same kind; the ratio of two abstract numbers.

4. A direct ratio; an inverse ratio.

5. Proportion; proportionals; a continued proportion.

6. An antecedent; a consequent; the extremes; the means.

7. A mean proportional; a third proportional.

8. An inequality; larger-smaller inequalities; greater-less inequalities.

9. A product where the multiplier is an incommensurable number; a quotient where the elements are incommensurable.

10. A lemma; an indirect proof.

State, with any necessary explanations or illustrations:

11. The axioms of larger-smaller inequality.

12. The axioms of greater-less inequality.

13. How to find a missing proportional.

14. The theorems that apply to proportions involving concrete numbers.

Prove that:

15. The product of the extremes of a proportion equals that of the means.

16. The factors of two equal products form a proportion.

17. A proportion may be written by alternation; by inversion; by composition; and by division.

18. The terms of a proportion may be multiplied or divided by the like terms of another proportion.

19. The terms of a proportion may be multiplied or divided by the same number.

20. The antecedents of a proportion may be multiplied or divided by the same number, and so may the consequents.

21. The first and second terms of a proportion may be multiplied or divided by the same number, and so may the third and fourth terms.

22. The terms of a proportion may be raised to the same power, and the same root may be taken of them.

23. In a continued proportion the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

24. Two rectangles of equal altitude are as their bases.

25. Variation by a fixed law is expressed by an equation.

26. A proportion is an expression of direct variation.

27. Addition is a commutative and an associative operation with incommensurable, as with commensurable, numbers.

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28. What operations may, and what may not, be performed on the two members of a larger-smaller inequality? of a greater-less inequality?

29. State and prove the associative and commutative principles of multiplication, with incommensurable numbers.

30. The sum of the greatest and least of four positive proportionals is greater than the sum of the other two.

What is the law if some of the proportionals be negative?

31. If x, y be such numbers that $a + x : b + y = a : b$, then $x : y = a : b$.

32. The sum of a real positive number and its reciprocal cannot be less than 2.

33. If $x : y = (x + z)^2 : (y + z)^2$, z is a mean proportional between x and y.

34. If $a : b = c : d$ and b be a mean proportional between c and d, then c is a mean proportional between a and b.

35. If $a \propto b^2$, $b \propto c^3$, $c \propto d^5$, then $a \propto d^{30}$.

36. If $x \propto y^{-2}$, and $x = 1/2$ when $y = 4$, what is x when $y = 2/3$?

37. If $a : b = b : c = c : d$, then $a : d = a^3 : b^3$.

VI. POWERS AND ROOTS.

The words *power, root, base, exponent, root-index* are defined in I, § 5. A root-index is always assumed to be a positive integer; but an exponent may be any number whatever.

§ 1. THE BINOMIAL THEOREM.

Theor. 1. *If a binomial be raised to any positive integer power, that power is symmetric as to the two terms of the binomial, and consists of a series the number of whose terms is one greater than the exponent of the binomial; and the successive terms of this series are the products of three factors:*

1. *Coefficients that come from the exponent of the binomial: the first, one; the second, the exponent; the third, the product of the second by half the-exponent-less-one; the fourth, the product of the third by a third of the-exponent-less-two; and so on.*

2. *Falling powers of the first term, beginning with that power whose exponent is the exponent of the binomial.*

3. *Rising powers of the second term, beginning with the zero-power.*

Let a, b be any two numbers, then the theorem is written

$$(a+b)^n = a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2$$
$$+\tfrac{1}{6}n(n-1)(n-2)a^{n-3}b^3 + \cdots + b^n,$$

and it is proved by *induction*, as shown below.

(*a*) *The law is true for the second power.*

For $\because (a+b)^2 = a^2 + 2ab + b^2$, [multiplication.

$\therefore (a+b)^n = a^n + na^{n-1}b + \cdots + b^n$, when $n = 2$.

(*b*) *If the law be true for any one power it is true for the next higher power.*

For, assume $(a+b)^k = a^k + ka^{k-1}b + \frac{1}{2}k(k-1)a^{k-2}b + \cdots + b^k$,
and multiply both members of this equation by $a+b$,
then $(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{1}{2}(k+1)ka^{k-1}b^2 + \cdots + b^{k+1}$
a series of the same form as that for the kth power, $k+1$ taking the place of k, and conforming to the law of development.

(c) *The law is true, whatever the power.*

For $\because$ it is true for the second power, [(a)

$\therefore$ it is true for the third power; [(b)

and $\because$ it is true for the third power, [above

$\therefore$ it is true for the fourth power, and so on. Q.E.D.

QUESTIONS.

1. In any power of $(a-b)$ what terms are negative? In what powers is the last term negative? positive?

Expand:

2. $(x+y)^5$. 3. $(a-4)^4$. 4. $(a+b)^6$. 5. $(2x-y)^3$.
6. $(x^2+3y)^4$. 7. $(a-2bc)^4$. 8. $(a\pm b)^6$. 9. $(m-n)^7$.
10. $[3/(m+n)]^4$. 11. $[-2/(x-y)]^3$. 12. $(\frac{1}{2}x+2y)^5$.
13. $(2x/3-3/2x)^6$. 14. $(\frac{1}{2}x+\frac{1}{3}a)^4$. 15. $(4ax+3y-1)^3$.
16. $(3c+3d)^4$. 17. $(x+y-4z)^3$. 18. $(x-y+4z)^3$.
19. $[(2a-b)+(c-d)]^2$. 20. $[(2x+y)-(x-2y)]^4$.
21. A proof by induction consists of three steps: name them.
22. In the expansion of $(a+b)^{50}$, and of $(a-b)^{50}$, write the literal part of the third term; of the 50th term; of the 51st.
23. What term of $(a+b)^n$ is

$$n(n-1)\cdots(n-r+1)\cdot a^{n-r}\cdot b^r/r!? \quad [r! \equiv 1\cdot 2\cdot 3\cdots r.$$

How may the last factor in the numerator of a coefficient be found from the number of the term? in the denominator?

24. Of what term of $(a+b)^{50}$ is $50\cdot 49\cdot 48\cdots 43/8!$ the coefficient? $50\cdot 49\cdot 48\cdots 9/42!$?
25. What other term has the same coefficient as the 49th? the 50th? the 41st? What is the general principle?

Make use of the general formula in ex. 23 to write:

26. The fourth term of $(x-5)^{12}$; the 7th term; the 12th.
27. The 12th term of $(1-\frac{1}{2}x^2)^{14}$; the third term; the 8th.
28. The middle term of $(a/x+x/a)^{10}$; the third; the 7th.
29. In $(x+y)^{10}$ the sum of the coefficients of the odd terms equals the sum of the coefficients of the even terms.
30. What term of $(x+1/x)^{2n}$ is free from x?

§ 2. FRACTION POWERS.

A *fraction power* of a number is either a root of the number or some integer power of a root. The record of the operation begins by naming the base, then the number of factors it is resolved into, then the number of such factors that are multiplied together. The two numbers last named appear as the denominator and the numerator of a simple fraction.

E.g., $64^{2/3} \equiv (\sqrt[3]{64})^2 = 4 \cdot 4 = 16$, wherein 64 is resolved into the three equal factors 4, 4, 4, and the product of two of them is taken.

So, $64^{-2/3} \equiv (\sqrt[3]{64})^{-2} = 1/4^2 = 1/16$, wherein 64 is resolved into the three equal factors 4, 4, 4, and two of these factors are used in partition.

Note the new use of the fraction form 2/3: As an exponent, it means that 64 is *resolved into three equal factors,* and *two of them are multiplied together;* in the ordinary usage, some unit is *divided into three equal parts,* and *two of them are added together.* Later it appears that these fraction exponents are subject to the laws already established for fractions; but this must not be assumed without proof.

Integer powers and fraction powers are classed together as *commensurable powers;* incommensurable powers appear later.

The value of a fraction power is often ambiguous.

E.g., $100^{1/2} = \pm 10$; $9^{-3/2} = \pm 1/27$; $(a^2)^{1/2} = \pm a$; $(a^4)^{3/4} = \pm a^3$.

Different powers of a base are *in the same series* if they be integer powers of the same root. An integer power of a base belongs to all series alike.

E.g.,	9^{-1},	$9^{-1/2}$,	9^0,	$9^{1/2}$,	9^1,	$9^{3/2}$,	9^2, $\cdots$	are the
	-2d,	-1st,	0th,	1st,	2d,	3d,	4th, $\cdots$	powers of $\sqrt{9}$,
i.e.,	1/9,	$^-1/3$,	1,	$^-3$,	9,	$^-27$,	81, $\cdots$,	powers of $^-3$
and	1/9,	1/3,	1,	3,	9,	27,	81, $\cdots$,	powers of $^+3$

When several powers of the same base occur together, they are assumed to be all taken in the same series.

Powers that have the same exponent are *like powers.*

E.g., $\sqrt{a}$, $\sqrt{b}$, $\sqrt{ab}$; a^2, b^2, $\overline{ab}^2$; 2^n, 3^n, 6^n; $x^{-3/4}$, $y^{-3/4}$, $z^{-3/4}$.

QUESTIONS.

Explain the meaning of each part of the expressions below.

1. $-27^{2/3}$. 2. $-27^{-2/3}$. 3. $-27^{4/3}$. 4. $81^{3/4}$.

5. $81^{-1/4}$. 6. $243^{2/5}$. 7. $-243^{7/5}$. 8. $243^{3/5}$.

9. $-243^{1/5}$. 10. $243^{-4/5}$. 11. $81^{-1/2}$. 12. $1728^{2/3}$.

13. What operation is indicated by the denominator of a simple fraction? what by the numerator? what by the terms of a fraction exponent? Can it be assumed that fractions, as exponents, are operated with just as other fractions?

Express with fraction exponents:

14. $\sqrt{x^5}$. 15. $\sqrt[3]{a^4}$. 16. $\sqrt[5]{a^5b^2c}$.

17. $(\sqrt[3]{y^2})^4$. 18. $4\sqrt{abc^3/x}$. 19. $\sqrt{(xyz/b^2)}$.

20. $(3\sqrt[3]{b^2cx})^5$. 21. $\sqrt[4]{2a}\sqrt{(x/y)}$. 22. $\sqrt[5]{(x+y)/(x-y)^2}$.

Express with radical signs:

23. $(a^{5/2})^3$. 24. $(b^{-3/4})^{2/5}$. 25. $(c-d)^{2/9}$.

26. $(x^3y^{1/2})^{-2/3}$. 27. $(5ax^{-2/3})^{-2/3}$. 28. $3/x^2y^{3/4}/4m^{1/5}$.

29. $a^{2/3}/b^{-3/4}$. 30. $8a^{-1/3}+b^{-2/5}$. 31. $x^{1/2}+y^{2/3}/z^{3/4}$.

Express with positive exponents:

32. x^{-3}. 33. $6a^{-3}x^{-2}$. 34. $a^{-3}b^{-1}c^4$.

35. $\dfrac{4x^{1/2}y^{-2/3}}{x^{2/3}y^{-3/4}}$. 36. $\dfrac{2x^{-1}y}{3^{-2}a^2x^{-2}}$. 37. $\dfrac{a^{-4}(x-y)^{-3/4}}{a^{-2}(x+y)^{-3/4}}$.

38. $\dfrac{x^{-2/3}y^{-n}}{a^{-2/3}b^{-4}}$. 39. $\dfrac{x^{-2/3}+y^{-n}}{a^{-2/3}+b^{-4}}$. 40. $\dfrac{x^{-2/3}-y^{-n}}{a^{-2/3}-b^{-4}}$.

41. Is the square root of an incommensurable number a commensurable or an incommensurable power? what is $2^{\sqrt{2}}$?

42. What kind of powers give rise to two values? what powers of these powers are alike in both series?

Find the first five powers of $16^{1/2}$ in both series.

Find the value, or values, of:

43. $27^{-2/3}-27^{1/3}+(-27)^{2/3}$. 44. $25^{1/2}+25^{-3/2}+25^0$.

45. $16^{3/4}+16^{1/4}-16^{-1/4}-16^{-3/4}$. 46. $8^{-1/3}+8^{-2}-8^{2/3}+8^{-2/3}$.

47. $32^{1/5}-32^{2/5}+32^{-3/5}+32^{-4/5}$. 48. $36^{1/2}-36^{3/2}+36^{-5/2}$.

A COMMENSURABLE POWER OF A COMMENSURABLE POWER.

THEOR. 2. *A commensurable power of a commensurable power of a base is that power of the base whose exponent is the product of the two given exponents.*

Let m, n be any two commensurable numbers, and A any base; then $(A^m)^n = A^{mn}$.

(*a*) *m, n integers, positive or negative.* [I, th. 11.

(*b*) *m, n reciprocals of positive integers,* $1/p$, $1/q$.

For, resolve A into p equal factors B, B, $\cdots$, commensurable or incommensurable, so that $B = A^{1/p}$,

and resolve B into q equal factors C, C, $\cdots$, so that $C = (A^{1/p})^{1/q}$;

then $\because$ A is thus resolved into pq equal factors C, C, $\cdots$,

$\therefore (A^{1/p})^{1/q} = A^{1/pq}$. Q.E.D. [df. fract. pwr.

(*c*) *m, n, simple fractions,* p/q, r/s, and q, s *positive.*

For, resolve A into q equal factors B, B, $\cdots$ and take p of them, and resolve B into s equal factors C, C, $\cdots$ and take r of them; then $\because$ A is thus resolved into qs equal factors C, C, $\cdots$ and pr of them are taken,

$\therefore (A^{p/q})^{r/s} = A^{pr/qs}$. Q.E.D. [df. fract. pwr.

And so if three or more powers be taken in succession.

COR. 1. $(A^m)^{1/n} = A^{m/n}$.

COR. 2. $(A^n)^{1/n} = (A^{1/n})^n = A$.

EQUAL FRACTION POWERS.

THEOR. 3. *The value of a fraction power of a base is the same, whether the fraction exponent be in its lowest terms or not.*

For, let k, p, q be any positive integers, A any base,

then $A^{kp/kq} = \{[(A^{1/q})^{1/k}]^k\}^p$ [th. 2.

$= (A^{1/q})^p$ [th. 2 cr. 2

$= A^{p/q}$. Q.E.D. [df. fract. pwr.

QUESTIONS.

1. Explain the proof of theor. 2, and tell how it applies when A is not a perfect power of the pqth degree.

Find the values of:

2. $(8^{3/8}a^{-3})^{2/3}$. 3. $4(x^{-2/3})^{3/2}$. 4. $(64y^6)^{-5/6}$.

5. $\sqrt[6]{}(a^3bc\sqrt[5]{}a^3bc)^5$. 6. $(9^{1/2})^3$. 7. $(125^{-2/3})^{-3/2}$.

8. $[(4x^2-12x+9)^{1/2}]^3$. 9. $(16a^{-4}/81b^3)^{-3/4}$.

10. $(9x^4/25y^3)^{-3/2}$. 11. $(256/625)^{-3/4}$.

12. $\sqrt[4]{}(a\sqrt{}b/\sqrt[3]{}ab)^3$. 13. $\sqrt[5]{}x^2y^3/x^{-1/2}y^{1/2}$.

14. Resolve 64 into six equal factors and indicate the continued product of twenty-five of them; then resolve 64 into two equal factors, resolve the product of five of these factors into three equal factors, and indicate the prime factors in the product of five of them: hence show that $(64^{5/2})^{5/3}=64^{25/6}$.

15. In the expression $A^{kp/kq}$, can the exponent kp/kq be replaced by the equal fraction p/q without further proof?

16. Resolve 729 into three equal factors and take the product of two of them; then resolve 729 into six equal factors and take the product of four of them: hence show that $729^{2/3}=729^{4/6}$.

Simplify the radicals:

17. $x^{3/6}$. 18. $(a+b)^{-6/8}$. 19. $(x-y)^{15/21}$. 20. $a^{17/-51}$.

21. $\sqrt[4]{}a^2$. 22. $\sqrt[6]{}(a-b)^4$. 23. $\sqrt[12]{}p^2q^{-3}$. 24. $\sqrt[15]{}(x^3/y^5)$.

25. $(x\sqrt{}y)^4$. 26. $(x^{-1/2}/y^{-1/2})^6$. 27. $(a^{2/3}/\sqrt[3]{}a^2)^9$. 28. $(8a^{-3})^{-2/6}$.

29. How can fraction powers of different degrees be reduced to powers of like roots? are their values changed thereby?

Reduce to powers of the twelfth root:

30. $x^{1/3}, 3x^{1/4}, a^{1/2}$. 31. $(x+y)^{2/3}, y^{3/4}$. 32. $-b^{-1/3}, c^2, a^{5/6}$.

33. $x^{-1}, y^{7/12}, z^{-1/6}$. 34. $(x+y)^{-2/3}, y^{-3/4}$. 35. $b^{1/3}, c^{-2}, a^{-5/6}$.

36. $\sqrt{}x, \sqrt[3]{}y, \sqrt[4]{}z^3$. 37. $(\sqrt{}x\sqrt{}y)^{2/3}, z^{-2/3}$. 38. $(a^{1/3})^{-1/4}, (a^{-1/3})^{1/4}$.

Reduce to powers of like roots:

39. $\sqrt[m]{}x, \sqrt[n]{}y$. 40. $\sqrt[m]{}a^n, \sqrt[n]{}a^m, \sqrt[2mn]{}a$. 41. $\sqrt{}2, \sqrt[3]{}3$.

42. $\sqrt{}ab, \sqrt[n]{}c$. 43. $x^{1/2}, y^{2/3}$. 44. $x^{-1/2}, y^{-2/3}$.

PRODUCT OF COMMENSURABLE POWERS OF THE SAME BASE.

Theor. 4. *The product of two or more commensurable powers of a base is that power of the base whose exponent is the sum of the exponents of the factors.*

Let m, n be two commensurable numbers, and A any base;

then $A^m \cdot A^n = A^{m+n}$.

(*a*) *m, n integers, positive or negative.* [I, th. 10.

(*b*) *m, n simple fractions, p/q, r/s, and q, s positive.*

For $A^{p/q} \cdot A^{r/s} = A^{ps/qs} \cdot A^{qr/qs}$ [th. 3.

$= (A^{1/qs})^{ps} \cdot (A^{1/qs})^{qr}$ [df. fract. pwr.

$= (A^{1/qs})^{ps+qr}$ [(*a*).

$= A^{ps/qs+qr/qs}$ [th. 2.

$= A^{p/q+r/s}$. Q.E.D. [th. 3.

And so for three or more powers.

Cor. $A^m/A^n = A^{m-n}$.

PRODUCT OF LIKE COMMENSURABLE POWERS OF DIFFERENT BASES.

Theor. 5. *The product of like commensurable powers of two or more bases is the like power of their product.*

Let n be any commensurable number, and A, B, C, $\cdots$ any bases;

then $A^n \cdot B^n \cdot C^n \cdots = (A \cdot B \cdot C \cdots)^n$.

(*a*) *n an integer, positive or negative.* [I, th. 12.

(*b*) *n a simple fraction, p/q.*

For $\because A^p \cdot B^p = (A \cdot B)^p$, [(*a*).

and $A^p = (A^{p/q})^q$, $B^p = (B^{p/q})^q$, [th. 2.

$\therefore (A^{p/q})^q \cdot (B^{p/q})^q = (A \cdot B)^p$,

and $\because (A^{p/q})^q \cdot (B^{p/q})^q = (A^{p/q} \cdot B^{p/q})^q$, [(*a*).

$\therefore (A^{p/q} \cdot B^{p/q})^q = (A \cdot B)^p$;

$\therefore A^{p/q} \cdot B^{p/q} = (A \cdot B)^{p/q}$. [III, ax. 7.

Cor. $A^n/B^n = (A/B)^n$.

The qth roots involved in the last statement must be used with some caution as to the signs: the equation means that the real positive roots, if any, are equal, and that all the roots have the same magnitude.

QUESTIONS.

1. Does the proof of theor. 4 differ from that given, when A is an incommensurable number?

2. Prove theor. 4 for the product of three or more powers.

3. Assume $A^m/A^n \neq A^{m-n}$, and multiply both members by A^n: if the result be contradictory to theor. 4, what is proved? What is such a method of proof called?

Write the expressions below in their simplest forms, using only positive exponents:

4. $81^{3/4}\cdot 81^{-1/2}$. 5. $81^{1/2}/81^{-1/4}$. 6. $256^{5/8}\cdot 256^{-3/4}/256^{1/2}$.

7. $ab^{1/2}c\cdot a^{-1/2}bc$. 8. $ab^{1/2}c/a^{-1/2}bc$. 9. $x^{3/8}y^{1/4}z^{1/2}\cdot x^{-3/8}y^{-1/4}$.

10. $[\{(a^{-m})^{-n}\}^p]^q : [\{(a^m)^n\}^{-p}]^{-q}$. 11. $x^{3/8}y^{1/4}z^{1/2}/x^{-3/8}y^{-1/4}$.

12. $\left(\dfrac{ay}{x}\right)^{1/2}\cdot\left(\dfrac{bx}{y^2}\right)^{1/3}\cdot\left(\dfrac{y^2}{a^2b^2}\right)^{1/4}$. 13. $\dfrac{(x-y)^{-3}\cdot(x+y)^2}{(x+y):(x-y)^{-1}}$.

14. Prove that $A^{-p/q}\cdot B^{-p/q}\cdot C^{-p/q} = (A\cdot B\cdot C)^{-p/q}$.

15. How many signs has $A^{1/q}$, or $B^{1/q}$, when q is odd? even? In each case what is the sign of the product $A^{1/q}\cdot B^{1/q}$? Show that in any case, a value of $(A\cdot B)^{1/q}$ has the same sign.

16. Multiply the product of two of the three equal factors of 8 by the product of two of the three equal factors of 125, and compare the result with the product of two of the three equal factors of the product $8\cdot 125$: write the conclusion, using fraction exponents.

Find the values of:

17. $81^{1/4}\cdot 256^{1/4}$. 18. $64^{2/3}\cdot 125^{2/3}$. 19. $64^{-2/3}/125^{-2/3}$.

20. $(ax)^{1/n}y^{-1/n}$. 21. $\sqrt{x^3}\cdot\sqrt{y^3}$. 22. $\sqrt{x^3}/\sqrt{y^3}$.

23. $\left(\dfrac{a-b}{a^2+ab}\right)^3\cdot\left(\dfrac{a^2-b^2}{a^2-ab}\right)^3$. 24. $\left(\dfrac{m^2-n^2}{x-y}\right)^2 : \left(\dfrac{n-m}{x-y}\right)^2$.

25. If $a^b = b^a$, then $(a/b)^{a/b} = a^{a/b-1}$; and if $a=2b$, then $b=2$.

§ 3. RADICALS.

A *radical* is an indicated root of a number. There may be a coefficient; and then the whole expression is called a radical, and the indicated root is the *radical factor.*

Any expression that contains a radical is a *radical expression.*

A radical is *rational* if the root can be found and exactly expressed in commensurable numbers; *irrational,* if the root cannot be so found. It is *real* if it do not involve the even root of a negative; *imaginary,* if it involve such root.

An expression that contains an irrational radical is a *surd.*

E.g., $\sqrt[8]{256}$, $\sqrt[3]{8}$, $\sqrt[3]{{}^-8}$, $\sqrt[5]{a^5}$, $\sqrt{(a^2+2ab+b^2)}$ are radicals with the rational values ${}^\pm 2$, 2, ${}^-2$, a, ${}^\pm(a+b)$;

but $\sqrt{x}$, $\sqrt[2]{a^3}$, $\sqrt[3]{a^4}$, $\frac{4}{9}a\cdot a^{-1/4}$, $\frac{5}{2}(a^2+b^2)^{1/2}$ are irrational; and, while all these radicals are real,

$\sqrt{{}^-1}$, $\sqrt{{}^-a^3}$, $\sqrt[4]{{}^-2a^3}$, $({}^-a)^{3/4}$, $a+b\sqrt{{}^-1}$ are imaginary.

Roots of rational bases, and integer powers of such roots, with rational coefficients, if any, are *simple* radicals.

The *degree* of a simple radical is shown by its root index.

A simple radical is *quadratic, cubic, quartic,* (biquadratic) $\cdots$ when the root index is 2, 3, 4, $\cdots$

E.g., $\frac{3}{2}(a^2+b^2)^{1/2}$, $3ab^2\cdot\sqrt[3]{(a^2-bc^3)}$, $a^2\cdot a^{3/4}$, are simple quadratic, cubic, and quartic surds in their simplest forms;

but ${}^\pm\sqrt{a^3}$, $\sqrt[3]{a^4}$, $\sqrt{8}$, $\sqrt{{}^-8}$, $\frac{3}{2}(a^2c^2+b^2c^2)^{1/2}$, are simple radicals not in their simplest forms; they may be written:

$${}^\pm a\sqrt{a},\quad a\sqrt[3]{a},\quad {}^\pm 2\sqrt{2}.\quad {}^\pm 2\sqrt{{}^-2}.\quad {}^\pm\tfrac{3}{2}c(a^2+b^2)^{1/2}.$$

Two radicals are *like* (similar) if they have the same radical factor; they are *conformable* if they can be made like; *non-conformable* if they cannot be made like.

E.g., $2x(a^2+b^2)^{2/3}$, $8(x-y)(a^2+b^2)^{2/3}$ are like radicals, and $\sqrt{18}$, $\sqrt{32}$, $\sqrt{98}$ are conformable.

The sum of two non-conformable simple surds, or of a rational expression and a simple surd, is a *binomial surd;* the sum of three non-conformable simple surds, or of two such surds and a rational expression, is a *trinomial surd.*

QUESTIONS.

Are these radicals rational or irrational? real or imaginary?

1. $\sqrt{5}$. 2. $\sqrt[3]{343}$. 3. $\sqrt{-256}$. 4. $\sqrt[3]{-243}$.
5. $\sqrt[3]{(x^3-y^3)}$. 6. $\sqrt{(1-2\sqrt{x}+x)}$. 7. $\sqrt[4]{256}\sqrt{-1}$. 8. $\sqrt[6]{-216}$.
9. What power of a radical is sure to be rational?

Tell why the statements below are true:

10. $x^{7/3}=x^2\cdot x^{1/3}$. 11. $\sqrt[3]{-4}=-\sqrt[3]{4}$. 12. $\sqrt{-4}\neq-\sqrt{4}$.
13. $4\sqrt{3}=\sqrt{48}$. 14. $3\sqrt{-4}=\sqrt{-36}$. 15. $-3\sqrt{4}=\sqrt{36}$.
16. $\sqrt{(2/5)}=\sqrt{(10/25)}=\sqrt{(1/25)}\cdot\sqrt{10}=\pm\frac{1}{5}\sqrt{10}$.
17. $-3x\sqrt[3]{5a^2b}=\sqrt[3]{-27x^3}\cdot\sqrt[3]{5a^2b}=\sqrt[3]{-135a^2bx^3}$.
18. $\sqrt{420a^3}=\pm2a\sqrt{105a}$. 19. $\sqrt[3]{(5/48)}=\frac{1}{12}\sqrt[3]{180}$.

Reduce the radicals below to their simplest forms.

20. $\sqrt{288}$. 21. $\sqrt{-169}$. 22. $\sqrt[3]{16(a+b)}$. 23. $\sqrt[5]{729}$.
24. $81^{-3/4}$. 25. $49^{5/2}$. 26. $500^{7/3}$. 27. $900^{24/96}$.
28. Change the quadratic surds on the opposite page to quartic surds, and the cubic surds to sextic surds.
29. $\sqrt{20}$, $\sqrt{45}$, $\sqrt{(4/5)}$ are conformable surds.
30. Change $\sqrt{2}$, $\sqrt[3]{3}$ to surds of the same degree.

If possible make like the radicals below, and reduce the expressions to their simplest form:

31. $\sqrt[4]{32}+6\sqrt[4]{2}$. 32. $\sqrt{3}+\sqrt{\frac{1}{3}}$. 33. $x^{1/3}+x^{1/2}$. 34. $\sqrt{x}-\sqrt[3]{y}$.
35. $5\sqrt{98x}+10\sqrt{2x}$. 36. $(3a^2b)^{1/2}-(27a^2b)^{1/2}$.
37. $\sqrt{(5x-5)}-\sqrt{(2x-2)}$. 38. $\sqrt[3]{(8a^3b+16a^4)}-\sqrt[3]{(b^4+2ab^3)}$.
39. $(36a^2y)^{1/2}-(25y)^{1/2}$. 40. $2/x^{-3/2}-\frac{1}{2}y\sqrt{x}$.
41. Separate 592704 into its prime factors: which of them occur three times? what is the value of $\sqrt[3]{592704}$?
42. So, $\sqrt{78400}$; $\sqrt[4]{50625}$; $\sqrt{27225}$; $\sqrt[4]{3111696}$.
43. Are $\sqrt{(a-b)}$ and y^2 rational or irrational?
Replace a by 22, b by 6, y by $\sqrt[3]{2}$: what changes are made?
Can a literal expression be imaginary and its numerical value be real, or the reverse?

OPERATIONS ON RADICALS.

Prob. 1. To reduce a radical to its simplest form.

Resolve the number whose root is sought into two factors: one the highest possible perfect power of the same degree as the radical, and the other an entire number;

write the root of the first-named factor as a coefficient before the indicated root of the other. [th. 5.

E.g., $\sqrt[3]{48a^3b^4} = \sqrt[3]{(8a^3b^3 \cdot 6b)} = 2ab\sqrt[3]{6b}$.

Prob. 2. To free a radical from coefficients.

Raise the coefficient to a power whose exponent is the root-index of the radical;

multiply this power by the expression under the radical sign, and put the same radical sign over the product. [th. 5.

E.g., $2ab\sqrt[3]{6b} = \sqrt[3]{(8a^3b^3 \cdot 6b)} = \sqrt[3]{48a^3b^4}$.

So, $a\sqrt[n]{(b^m - c^p)} = \sqrt[n]{[a^n \cdot (b^m - c^p)]} = \sqrt[n]{(a^nb^m - a^nc^p)}$.

Prob. 3. To reduce radicals to the same degree.

Write the radicals as fraction powers; [df. fract. pwr

reduce the fraction exponents to equivalent fractions having a common denominator; [th. 3.

restore the radical signs, using the common denominator as the root-index and the new numerators as exponents.

E.g., $ax,\quad \sqrt[3]{by},\quad \sqrt[5]{(b+c)} = ax,\quad (by)^{1/3},\quad (b+c)^{1/5},$

$= (ax)^{30/30},\quad (by)^{10/30},\quad (b+c)^{6/30},$

$= \sqrt[30]{(ax)^{30}},\quad \sqrt[30]{(by)^{10}},\quad \sqrt[30]{(b+c)^6}.$

Prob. 4. To add radicals.

Reduce the radicals to their simplest form; [pr. 1.

add like radicals by prefixing the sum of their coefficients to the common radical factor; [add. incom. num.

write unlike radicals in any convenient order.

E.g., $3\sqrt{8} + 5\sqrt{2} - 10\sqrt{32} = 6\sqrt{2} + 5\sqrt{2} - 40\sqrt{2} = -29\sqrt{2}$.

So, $a\sqrt[3]{b} + a^2\sqrt[3]{b^4} - a^3\sqrt[3]{b^7} = a\sqrt[3]{b} + a^2b\sqrt[3]{b} - a^3b^2\sqrt[3]{b}$

$= (a + a^2b - a^3b^2)\sqrt[3]{b}$.

QUESTIONS.

Reduce the radicals below to their simplest forms:

1. $125^{1/2}$. 2. $567^{1/2}$. 3. $392^{1/2}$. 4. $1008^{1/2}$.
5. $216^{1/3}$. 6. $72^{1/3}$. 7. $162^{1/3}$. 8. $48^{1/4}$.
9. $160^{1/5}$. 10. $(11\frac{13}{16})^{1/3}$. 11. $(6\frac{21}{64})^{1/3}$. 12. $(10\frac{1}{8})^{3/4}$.
13. $2500^{1/4}$. 14. $\sqrt[3]{296352}$. 15. $\sqrt{147x^{-3}yz^2}$. 16. $\sqrt[3]{56a^4b^5c^6}$.
17. $\sqrt[4]{112a^{-5}b^{-2}c^3}$. 18. $\sqrt[6]{64a^8b^{-3}c^4}$. 19. $\sqrt[8]{16a^4b^4c^{12}}$.
20. $\sqrt{50a^4b^5c^6d^7}$. 21. $\frac{1}{2}\sqrt{(3/7)}$. 22. $(a+b)\sqrt{[(a-b)/(a+b)]}$.
23. $2\sqrt{\dfrac{ab^2}{4(a+x)}}$. 24. $\dfrac{2a}{3x}\sqrt[3]{\dfrac{27x^4}{a^2}}$. 25. $\dfrac{a}{b}\sqrt[n]{\dfrac{b^2}{a^{n-2}}}$.
26. $(a/b)\sqrt[p]{(b^{q+1}/a^{p-1})}$. 27. $(a^{-1/2}/4c^2)^{-2}$. 28. $\sqrt[4]{(x^{-2/3}y^{1/2})^3}$.
29. $\sqrt{(72a^2b-72b+18a^{-2}b)}$. 30. $\sqrt[3]{[x^4y^{-1}-xy^2-3x^2(x-y)]}$.
31. $3\sqrt{147}-\frac{7}{3}\sqrt{(1/3)}-\sqrt{(1/27)}$. 32. $5\sqrt{24}-2\sqrt{54}-\sqrt{6}$.
33. $\dfrac{a-b}{x-y}\sqrt{\dfrac{x^2z+2xyz+y^2z}{a+b}}$. 34. $\dfrac{a^3-b^3}{a+b}\sqrt{\dfrac{3a^2+6ab+3b^2}{9(a^2-b^2)}}$.

Free the radicals below from coefficients:

35. $6\sqrt{5}$. 36. $2\sqrt{x}$. 37. $2x\sqrt{2}$. 38. $4a\sqrt{5b}$.
39. $4\sqrt[3]{6}$. 40. $5a\sqrt[3]{y}$. 41. $\frac{3}{2}\sqrt{9^{1/3}}$. 42. $\frac{1}{2}\sqrt{2b}$.
43. $\dfrac{a-b}{a+b}\sqrt{\dfrac{a+b}{a-b}}$. 44. $\dfrac{x^3y^2}{z^2}\cdot\left(\dfrac{z^5}{x^5y^5}\right)^{1/2}$. 45. $\dfrac{3}{a}\left(1-\dfrac{b^2}{a^2}\right)^{3/2}$.
46. $a\sqrt[5]{a^{-2}b^2x^6y^2/xy}$. 47. $(x^2-y^2)^{3/2}\cdot\sqrt{[(x-y)/(x^2+2xy+y^2)]}$.

Reduce the radicals below to the same degree:

48. $a^{1/2}$, $a^{1/3}$. 49. $a^{1/3}$, $b^{1/4}$. 50. $3^{1/2}$, $4^{1/4}$. 51. $a^{1/5}$, $b^{2/3}$.
52. $\sqrt[3]{ab}$, $\sqrt[4]{ac}$, $\sqrt[5]{bc}$, $\sqrt[6]{(b+c)}$. 53. $x^{1/2}$, $x^{2/3}$, $x^{3/4}$, $x^{4/5}$, $x^{5/6}$.
54. $3x^{1/2}$, $2y^{2/3}$, $4z^{3/4}$, $5v^{4/5}$. 55. $\sqrt[10]{a^5}$, $\sqrt[6]{b^3}$, $\sqrt[12]{c^6}$, $\sqrt[2n]{d^n}$, $\sqrt[8]{e^4}$.

Which is the larger:

56. $(1/2)^{1/2}$ or $(2/3)^{2/3}$? 57. $\sqrt[5]{2}$ or $\sqrt[8]{3}$? 58. $\sqrt[3]{9}$ or $\sqrt[4]{18}$?
59. $(\frac{1}{2})^{1/2}$, $(\frac{1}{3})^{1/3}$, $(\frac{1}{4})^{1/4}$, $(\frac{1}{5})^{1/5}$? 60. $\sqrt{(a+b+c)}$ or $\sqrt{a}+\sqrt{b}+\sqrt{c}$?

Find the sum of:

61. $\sqrt{18}-\sqrt{8}$. 62. $6\sqrt{(3/4)}-3\sqrt{(4/3)}$.
63. $3\sqrt{(2/5)}+4\sqrt{(1/10)}$. 64. $2\sqrt[3]{(1/5)}+3\sqrt[3]{(1/40)}$.
65. $\sqrt{128}-2\sqrt{50}+7\sqrt{72}$. 66. $a^2b^{1/3}+2ab^{4/3}+b^{7/3}$.
67. $9\sqrt{80}-2\sqrt{125}-5\sqrt{245}+\sqrt{320}$.

PROB. 5. TO MULTIPLY RADICALS.

If all the radicals have the same base, add the fraction exponents of the factors for the exponent of the product.

If the bases be different, but the radicals be of the same degree, write their product under the common radical sign.

If the radicals be not of the same degree, make them like.

If there be coefficients, prefix their product to that of the radical factors. [V, th. 10.

E.g., $3\sqrt{8}\cdot 5\sqrt{2}\cdot {}^{-}10\sqrt{32} = -3\cdot 5\cdot 10\cdot\sqrt{(8\cdot 2\cdot 32)}$
$= -150\cdot\sqrt{512} = -2400\cdot\sqrt{2}$.

So, $ab^{1/3}\cdot a^2b^{4/3}\cdot a^{-3}b^{-7/3} = a^{1+2-3}\cdot b^{1/3+4/3-7/3} = b^{-2/3}$.

NOTE 1. The product and quotient of two conformable simple quadratic surds are rational; of two such non-conformable surds they are surds.

For in the first case, the surd factors occur in pairs in the product and vanish in the quotient; and in the other, they are single, and the square root cannot be taken.

E.g., $\sqrt{6}$ is conformable with $\sqrt{(2/3)}$, but not with $\sqrt{5}$,
and $\sqrt{(6\cdot 2/3)}$, $\sqrt{(6:2/3)}$, $\sqrt{(2/3:6)}$ are rational.
but $\sqrt{(6\cdot 5)}$, $\sqrt{(6:5)}$, $\sqrt{(5:2/3)}$ are surds.

NOTE 2. The square of a binomial quadratic surd is a surd.

E.g., if $\sqrt{a}$, $\sqrt{b}$ be non-conformable surds,
then is $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$, a surd.

PROB. 6. TO GET A POWER (OR ROOT) OF A RADICAL.

Multiply the exponent of the given radical by the exponent of the power sought. [th. 2.

E.g., $(3\cdot 8^{1/2})^3 = 27\cdot 8^{3/2} = 432\cdot 2^{1/2} = 432\sqrt{2}$.

So, $\sqrt[3]{(3\cdot\sqrt{8})} = \sqrt[3]{\sqrt{72}} = \sqrt[6]{72}$;

$(a^3\cdot\sqrt{b^7})^5 = a^{15}\cdot\sqrt{b^{35}} = a^{15}\cdot b^{17}\cdot\sqrt{b}$;

$(a^3\cdot b^{7/2})^{1/5} = a^{3/5}b^{7/10}$.

Two quadratic binomial surds are *conjugate* if they differ only in the sign of one term.

E.g., $a+\sqrt{b}, a-\sqrt{b}$; $10^{1/2}+3, 10^{1/2}-3$; $\sqrt{x}+\sqrt{y}, \sqrt{x}-\sqrt{y}$.

QUESTIONS.

Reduce the expressions below to their simplest forms:

1. $3\sqrt{2}\cdot 2\sqrt{3}$. 2. $8\sqrt{6} : 2\sqrt{2}$. 3. $5\sqrt{7}\cdot 2\sqrt{7}$.

4. $3^{1/3}\cdot 2^{1/2}$. 5. $4\sqrt{3}\cdot 3\sqrt{5}\cdot 5\sqrt[3]{2}$. 6. $2^{1/2}\cdot 3^{1/3}\cdot 4^{1/4}$.

7. $(\frac{3}{2})^{3/4} : (\frac{3}{4})^{2/3}$. 8. $a^{1/2}\cdot b^{1/3}c^{1/4}/a^{-1/2}b^{-2/3}c^{-3/4}$.

9. $3\sqrt{6}\cdot 2\sqrt{3}\cdot 4\sqrt{5} : 12\sqrt{10}$. 10. $\frac{1}{10}\cdot(\frac{5}{9})^{1/3} : (\frac{9}{5})\cdot(\frac{15}{18})^{1/2}$.

11. $5^{3/4}\cdot 4^{2/3}\cdot 3^{3/2} : 60^{5/6}$. 12. $(\frac{5}{14})\cdot(\frac{2}{3})^{2/3} : \frac{5}{21}(\frac{9}{4})^{-1/3}$.

13. $\sqrt{(a^2-b^2)} : \sqrt{(a-b)} : \sqrt[4]{(a-b)}$. 14. $(5+2\sqrt{2})\cdot(5-2\sqrt{2})$.

15. $\frac{2}{3}a\sqrt[3]{b^2}\cdot\frac{3}{4}b\sqrt{a^3}\cdot\frac{2}{5}a^{-1/2}b^{-1/3}$. 16. $(8\sqrt{2}+2\sqrt{3})\cdot(2\sqrt{2}+\sqrt{3})$.

17. $(4+\sqrt{2})\cdot(1-\sqrt{3})\cdot(4-\sqrt{2})\cdot(5-\sqrt{3})\cdot(1+\sqrt{3})\cdot(5+\sqrt{3})$.

18. $(a+b)^{1/m}\cdot(a+b)^{1/n}\cdot(a-b)^{1/m}\cdot(a-b)^{1/n}\cdot(a^2+b^2)^{(m+n)/mn}$.

19. $\sqrt{-a}\cdot\sqrt{-b}\cdot\sqrt[4]{-a}\cdot\sqrt[4]{-b}\cdot\sqrt[6]{-a}\cdot\sqrt[6]{-b}\cdot\sqrt[8]{-a}\cdot\sqrt[8]{-b}$.

20. $(\sqrt{2}+\sqrt{3}+\sqrt{5})\cdot(-\sqrt{2}+\sqrt{3}+\sqrt{5})$
$\cdot(\sqrt{2}-\sqrt{3}+\sqrt{5})\cdot(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

Find the powers and roots:

21. $(3\sqrt{3})^4$. 22. $(2\sqrt[3]{5})^6$. 23. $(\sqrt{2}-\sqrt{3})^2$.

24. $(\sqrt{10}-\sqrt{5})^2$. 25. $(3^{1/2}-3^{-1/2})^2$. 26. $(\sqrt{\frac{4}{3}}-\sqrt{\frac{3}{4}})^2$.

27. $(2^{1/3}-2^{-2/3})^3$. 28. $(3^{1/3}-3^{-1/3})^3$. 29. $(4^{1/3}+4^{-1/3})^4$.

30. $\sqrt[3]{(\frac{2}{3}x-\frac{3}{2}y)^2}$. 31. $[a^3b(a^3bc)^{1/5}]^{1/6}$. 32. $\sqrt[m]{-2^{1/m}a^mb^mc^{2m}}$.

33. $(a^{1/3}b^{-1/3}+a^{-1/3}b^{1/3})^3$. 34. $[(a+b)^{1/2}-(a-b)^{1/2}]^2$.

35. Find the first five powers of $\sqrt{^-1}$.

Are the even powers of $\sqrt{^-1}$ real or imaginary? are the fourth, eighth, twelfth powers positive or negative? the second, sixth, tenth powers?

Write the surds that are conjugate to:

36. $3-2\sqrt{x}$. 37. $\sqrt{3}-2\sqrt{x}$. 38. $\sqrt{(a+b)}+\sqrt{(a-b)}$.

39. $3+2\sqrt{5}$. 40. $\sqrt{3}+2\sqrt{5}$. 41. $x-3y^{-1/2}$. 42. $x^{1/2}-3y^{-1/2}$.

43. What terms of the square of $\sqrt{a}+\sqrt{b}$ are products of conformable factors? of non-conformable factors?

44. If the product of four simple quadratic surds be rational, the product of any two of them is conformable with that of the other two.

Two surds are *complementary* if their product be rational.

E.g., $a^{1/3}$, $a^{2/3}$; $5^{2/7}$, $5^{-2/7}$; $\sqrt{(a^2+b^2)}$, $\sqrt{(a^2+b^2)}$.

So, any two conjugate binomial surds are complementary; and three or more surds whose product is rational form a *group of complementary* surds.

E.g., $a+\sqrt{b}$, $a-\sqrt{b}$; $2+3\sqrt{-1}$, $2-3\sqrt{-1}$;

$b-\sqrt{(c+\sqrt{d})}$, $b+\sqrt{(c+\sqrt{d})}$, $b^2-c+\sqrt{d}$.

PROB. 7. TO REDUCE A FRACTION WITH A SURD DENOMINATOR TO AN EQUIVALENT FRACTION WITH A RATIONAL DENOMINATOR.

(*a*) *The denominator a monomial: multiply both terms of the fraction by some complement of the denominator.*

(*b*) *The denominator a simple binomial quadratic surd: multiply both terms of the fraction by the conjugate of the denominator.*

E.g., $a/b^{3/4}=a\cdot b^{1/4}/b$; $a/(\sqrt{b}-\sqrt{c})=a\cdot(\sqrt{b}+\sqrt{c})/(b-c)$.

(*c*) *The denominator a binomial quadratic surd containing a complex radical: multiply both terms of the fraction by a group of conjugate radicals that, taken together, are complementary to the denominator.*

E.g., $$\frac{a}{\sqrt{(b+\sqrt{c})}}=\frac{a\cdot\sqrt{(b+\sqrt{c})}}{b+\sqrt{c}}=\frac{a\cdot\sqrt{(b+\sqrt{c})}\cdot(b-\sqrt{c})}{b^2-c}.$$

(*d*) *The denominator any binomial surd: multiply the two fraction exponents of the denominator by the lowest common multiple of their denominators, and attach the products as exponents to the two bases;*

divide the sum, or difference, of the powers so found by the denominator, and multiply the numerator by the quotient.

E.g., to rationalize the denominator of $6^{1/2}/(2^{2/3}+3^{3/4})$:

then $\because$ 12 is the lowest common multiple of 3, 4,

and $12\cdot2/3=8$, $12\cdot3/4=9$,

and $\because (2^8-3^9):(2^{2/3}+3^{3/4})$

$$=2^{22/3}-2^{20/3}\cdot3^{3/4}+2^{18/3}\cdot3^{6/4}-\dots+2^{2/3}\cdot3^{30/4}-3^{33/4},$$

$$\therefore 6^{1/2}/(2^{2/3}+3^{3/4})=6^{1/2}\cdot(2^{22/3}-\dots-3^{33/4})/(2^8-3^9).$$

QUESTIONS.

Prove these surds complementary:

1. $c^{2/3}$, $c^{4/3}$. 2. $\sqrt{5}-6$, $\sqrt{5}+6$. 3. $x^{5/6}$, $x^{1/6}$. 4. $y^{3/4}$, $y^{-7/4}$.

5. $2x-y\sqrt{-1}$, $2x+y\sqrt{-1}$. 6. $a+b-\sqrt{c}$, $a+b+\sqrt{c}$.

7. $(4-3\sqrt{x})/(5+6\sqrt{2})$, $(4+3\sqrt{x})/(5-6\sqrt{2})$.

8. $\sqrt{2}+\sqrt{3}-\sqrt{5}$, $\sqrt{2}+\sqrt{3}+\sqrt{5}$, $\sqrt{6}$.

9. $\sqrt{a}-\sqrt{b}-\sqrt{c}$, $\sqrt{a}-\sqrt{b}+\sqrt{c}$, $a+b-c+2\sqrt{ab}$.

10. The product of any two conjugate quadratic binomial surds is rational: what other complement has such a surd besides its conjugate?

11. The product of two surds differing only in the sign of one term, but of higher degree than the second, is not rational.

Write three complements of each of the surds:

12. $m^{-3/2}$. 13. $(x^2)^{1/3}$. 14. $(a-b)^{2/5}$. 15. $ab^{-1/2}$.

Reduce to equivalent fractions with rational denominators:

16. $1/\sqrt{3}$. 17. $6/\sqrt{2}$. 18. $3\sqrt{8}/2\sqrt{2}$. 19. $2x/3y^{1/2}$.

20. $4x/3y^{3/4}$. 21. $(a/y)^{2/3}$. 22. $(m/n)^{-4/5}$. 23. $2/(\sqrt{3}+1)$.

24. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$. 25. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$. 26. $\dfrac{21}{\sqrt{10}-\sqrt{3}}$. 27. $\dfrac{1+\sqrt{5}}{\sqrt{5}-\sqrt{3}}$.

28. $\dfrac{(a+b)^{1/2}+(a-b)^{1/2}}{(a+b)^{1/2}-(a-b)^{1/2}}$. 29. $\dfrac{1}{a+\sqrt{[b+\sqrt{(c+\sqrt{d})}]}}$.

30. $\dfrac{\sqrt{2}\cdot(\sqrt{2}-3)}{(\sqrt{2}+8)\cdot(\sqrt{3}-\sqrt{5})}$. 31. $\dfrac{x+(x^2-1)^{1/2}}{x-(x^2-1)^{1/2}}$.

32. $\dfrac{x-(x^2-1)^{1/2}}{x+(x^2-1)^{1/2}}$. 33. $\dfrac{x-(x^2+1)^{1/2}}{x+(x^2+1)^{1/2}}$.

34. $\dfrac{(a+\sqrt{-1})^3-(a-\sqrt{-1})^3}{(a+\sqrt{-1})^2-(a-\sqrt{-1})^2}$. 35. $\dfrac{a\sqrt{(1-b^2)}-b\sqrt{(1-a^2)}}{\sqrt{(1-b^2)}+\sqrt{(1-a^2)}}$.

36. $\dfrac{\sqrt{6}-\sqrt{5}-\sqrt{3}+\sqrt{2}}{\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}}$. 37. $\dfrac{1+3\sqrt{2}-2\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

38. $(3+\sqrt{3})\cdot(3+\sqrt{5})\cdot(\sqrt{5}-2)/(5-\sqrt{5})/(1+\sqrt{3})=\frac{1}{5}\sqrt{15}$.

EQUATIONS THAT CONTAIN SURDS.

An equation that contains surds is *rationalized* when it is replaced by an equivalent equation free from surds.

E.g., the equation $x=\sqrt{2}$, i.e., $x=\overset{+}{\sqrt{}}2$ or $x=\overset{-}{\sqrt{}}2$,
when rationalized, becomes $x^2=2$.

So, if $\sqrt{x}+\sqrt{y}=0$:
then $x-y=0$; [mult. ea. mem. by $\sqrt{x}-\sqrt{y}$.
or $\sqrt{x}=-\sqrt{y}$, [ax. add.
and $x=y$. [squaring.

So, if $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$:
then $x+y-z+2\sqrt{xy}=0$, [mult. by $(\sqrt{x}+\sqrt{y}-\sqrt{z})$.
$2\sqrt{xy}=z-x-y$, [ax. add.
and $4xy=z^2+x^2+y^2+2xy-2zx-2zy$. [squaring.

So, if $x^{2/3}+x^{1/3}-1=0$:
then $x-2x^{1/3}+1=0$, [mult. by $(x^{1/3}-1)$.
$2x^{1/3}=x+1$, $8x=x^3+3x^2+3x+1$, [cubing.
and $x^3+3x^2-5x+1=0$.

So, if $\dfrac{x-\sqrt{x}-12}{x+\sqrt{x}-6}=\dfrac{\sqrt{x}+2}{4\sqrt{x}+1}$:
then $\dfrac{\sqrt{x}-4}{\sqrt{x}-2}=\dfrac{\sqrt{x}+2}{4\sqrt{x}+1}$, [red. first mem. to lowest terms.
$4x-15\sqrt{x}-4=x-4$, [clear. of fract.
and $x=5\sqrt{x}$, $x^2=25x$, $x=25$ or 0.

So, if $\sqrt{(3+x)}+\sqrt{x}=6/\sqrt{(3+x)}$:
then $3+x+\sqrt{(3x+x^2)}=6$, [clear of fract.
$\sqrt{(3x+x^2)}=3-x$,
and $3x+x^2=9-6x+x^2$, $9x=9$, $x=1$. [squaring.

So, if $x^{1/2}-[x-(1-x)^{1/2}]^{1/2}=1$:
then $[x-(1-x)^{1/2}]^{1/2}=x^{1/2}-1$, $x-(1-x)^{1/2}=x-2x^{1/2}+1$,
$(1-x)^{1/2}=2x^{1/2}-1$, $(1-x)=4x-4x^{1/2}+1$,
and $4x^{1/2}=5x$, $16x=25x^2$, $x=16/25$ or 0.

QUESTIONS.

Solve the equations:

1. $\sqrt{(3-x)}+6=7$. 2. $\sqrt{(x+4)}=4-\sqrt{x}$. 3. $y^{1/3}+5=11$.

4. $x+a=\sqrt{[a^2+x\sqrt{(b^2+x^2)}]}$. 5. $\sqrt[4]{(20-\sqrt{2x})}=2$.

6. $\dfrac{ax-1}{\sqrt{ax}+1}=4+\dfrac{\sqrt{ax}-1}{2}$. [red. first fract. to lower terms.

7. $\dfrac{\sqrt{3x}-\sqrt{3}}{\sqrt{2x}-\sqrt{2}}=\dfrac{x+3}{2}$. 8. $\dfrac{\sqrt{(4x+1)}+\sqrt{4x}}{\sqrt{(4x+1)}-\sqrt{4x}}=9$.

9. $x^{1/2}+(a+x)^{1/2}=2a(a+x)^{-1/2}$. 10. $(x^2-1)^{1/2}=8(x^2-1)^{-1/2}$.

11. $(x^2-3x+4)^{1/2}=x-3$. 12. $(8-4x)^{1/2}+(13-4x)^{1/2}=5$.

13. $18(2x+3)^{-1/2}=(2x-3)^{1/2}+(2x+3)^{1/2}$.

14. $3\sqrt{(x-\frac{5}{9})}+7\sqrt{(x+\frac{8}{49})}=10\sqrt{(x+.03)}$.

15. $\sqrt[3]{(\sqrt{3}+x\sqrt{7})}+\sqrt[3]{(\sqrt{3}-x\sqrt{7})}=\sqrt[6]{12}$.

16. $\sqrt{(a+x)}+\sqrt{(a-x)}=b[\sqrt{(a+x)}-\sqrt{(a-x)}]$.

Rationalize the equations:

17. $\sqrt{a}+\sqrt{b}+c=0$. 18. $3\sqrt{x}-2\sqrt{5y}=-1$.

19. $\sqrt{x}-\sqrt{y}=\sqrt{y}-\sqrt{x}$. 20. $-\sqrt[3]{x^2}-\sqrt[3]{x}+1=0$.

21. $\sqrt[3]{x^2}-\sqrt[3]{x}-1=0$. 22. $\sqrt[3]{x^2}-\sqrt[3]{x}+1=0$.

23. What effect is produced on the degree of an equation by rationalizing it?

24. Divide 18 into two parts whose squares shall be to each other as 25 to 16. In solving this problem four square roots are found: what is the effect if one of them be taken negative?

25. If to a certain number 22577 be added, the square root of the sum be taken, and from this root 163 be subtracted, the remainder is 237: what is the number?

26. The length of the side of a square whose area is 1 square inch less than that of a given square, increased by the side of a square whose area is 4 square inches more than that of the given square, equals the side of a square whose area is 5 square inches more than 4 times that of the given square: find the area of the given square.

§ 4. ROOTS OF POLYNOMIALS.

Evolution is an inverse operation: the work is an effort to retrace the steps taken in getting the power whose root is now sought; it is a process of trial, by progressive steps, like division and other inverse operations, and its success is established by raising the root to the required power and finding the result identical with the given polynomial.

SQUARE ROOT.

PROB. 8. TO FIND THE SQUARE ROOT OF A POLYNOMIAL.

Arrange the terms of the polynomial in the order of the powers of some one letter, a perfect square first;

take the square root of the first term of the polynomial as the first term of the root;

divide the remainder by double the root so found, and make the quotient the next term of the root and of the divisor;

multiply the complete divisor by this term and subtract the product from the dividend;

double the root found for a new trial divisor, and proceed as before.

The rule is based on the type form for the expansion of the square of a polynomial; when any complete divisor is multiplied by the new term of the root, and the product is subtracted from the last remainder, the whole root thus far found is thereby squared and subtracted from the polynomial.

E.g.,

$$\begin{array}{rl}
 & a^2+2ab+b^2+2ac+2bc+c^2\,(a+b+c. \\
 & \underline{a^2} \\
\overline{2a+b\,}) & 2ab+b^2 \\
\overline{2a+2b+c\,}) & \underline{2ac+2bc+c^2}
\end{array}$$

So,

$$\begin{array}{rl}
 & (3x^{-2}-ay^{2/3}+a^{-1}z^4 \\
 & 9x^{-4}-6ax^{-2}y^{2/3}+a^2y^{4/3}+6a^{-1}x^{-2}z^4-2y^{2/3}z^4+a^{-2}z^8 \\
 & \underline{9x^{-4}} \\
6x^{-2}-\overline{ay^{2/3}}) & \underline{-6ax^{-2}y^{2/3}+a^2y^{4/3}} \\
\overline{6x^{-2}-2ay^{2/3}+a^{-1}z^4}) & \underline{6a^{-1}x^{-2}z^4-2y^{2/3}z^4+a^{-2}z^8}
\end{array}$$

QUESTIONS.

Find the square root, preferably by detached coefficients, of:

1. $16x^2-40xy+25y^2$. 2. $1+2x+7x^2+6x^3+9x^4$.

3. $a^4-2a^3+2a^2-a+\frac{1}{4}$. 4. $x^{-4}-2x^{-2}+3-2x^2+x^4$.

5. $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{7}{4}$. 6. $\dfrac{a^2}{b^2}-\dfrac{2a}{b}+3-\dfrac{2b}{a}+\dfrac{b^2}{a^2}$.

7. $9x^2-30ax-3a^2x+25a^2+5a^3+\frac{1}{4}a^4$.

8. $4a^2-12ab+4ax+9b^2-6bx+x^2$.

9. $9a^6-12a^5-26a^4+44a^3+9a^2-40a+16$.

10. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$.

11. $\frac{4}{9}a^2x^2-\frac{4}{3}abx^2z+\frac{8}{3}a^2bx^2z^2+b^2x^2z^2-4ab^2x^2z^3+4a^2b^2x^2z^4$.

12. $4x^2+\frac{1}{9}x^{-2}+16x^{-4}-\frac{4}{3}+16x^{-1}-\frac{8}{3}x^{-3}$.

13. $a+4a^{2/3}+9a^{1/2}+4a^{5/6}-6a^{3/4}-12a^{7/12}$.

14. $1+4/x+10/x^2+20/x^3+25/x^4+24/x^5+16/x^6$.

15. $1-4/z+10/z^2-20/z^3+25/z^4-24/z^5+16/z^6$.

16. How many terms has the square of a binomial?

Supply a third term to a^2+2ab that shall make the resulting trinomial a perfect square; so, to a^2+b^2; to $2ab+b^2$.

Make a general rule for completing such a square.

Complete a square from each of the expressions:

17. x^2+6x. 18. x^2-4x. 19. x^2+5x. 20. x^2-7x.

21. x^4-3x^2. 22. x^6+8x^3. 23. y^8-12y^4. 24. $9z^2+12z$.

25. $16x^2-24x$. 26. $25x^4-20x^2$. 27. $9z^4-12z^3$. 28. $4x^2-10x$.

29. $(y^2+y-6)^2-3(y^2+y-6)$. 30. $(x+y)^2+2(x+y)$.

31. $(x^2+2x-1)^2+4x^2+8x-4$. 32. $(a-3y)^4+6(a-3y)^2$.

Find four terms of the square root of:

33. $1+x^2$. 34. x^2+1. 35. x^2-a^2. 36. a^2-x^2.

37. $1-x^2$. 38. x^2-1. 39. x^2+a^2. 40. a^2+x^2.

Show that the expressions below are perfect squares:

41. $(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3$
$\qquad -3(x^2-yz)\cdot(y^2-zx)\cdot(z^2-xy)$.

42. $4[(a^2-b^2)cd+(c^2-d^2)ab]^2+[(a^2-b^2)(c^2-d^2)-4abcd]^2$.

CUBE ROOT.

PROB. 9. TO FIND THE CUBE ROOT OF A POLYNOMIAL.

Arrange the terms of the polynomial in the order of the powers of some one letter, a perfect cube first;

take the cube root of the first term of the polynomial as the first term of the root;

divide the first term of the remainder by three times the square of the root found, and make the quotient the next term of the root;

to the divisor add three times the product of the first term of the root by the second term, and the square of the second term;

multiply the complete divisor by this term and subtract the product from the dividend;

take three times the square of the root found for a new trial divisor and proceed as before, treating the root so far found as the first term and the new term as the second.

The rule is based on the type-form for the expansion of the cube of a polynomial

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b;$$

and when any complete divisor is multiplied by the new term of the root and subtracted from the remainder, the whole root so far found is thereby cubed and subtracted from the polynomial; $3a^2$ is the trial, $3a^2 + 3ab + b^2$ the complete, divisor.

E.g.,

$$\begin{array}{rl}
 & (a + b + c \\
 & a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3ac^2 + 3b^2c + 3bc^2 + c^3 \\
 & a^3 \\
3a^2 + 3ab + b^2\,) & 3a^2b + 3ab^2 + b^3 \\
3a^2 + 6ab + 3b^2 + 3ac + 3bc + c^2\,) & 3a^2c + 6abc + 3ac^2 + 3b^2c + 3bc^2 + c^3
\end{array}$$

The rule given and illustrated under prob. 11 is perhaps the better rule for getting cube roots; the cubing of the root takes the place of the less familiar process of completing the divisor.

QUESTIONS.

Find the cube root, preferably by detached coefficients, of:

1. $1+6x+12x^2+8x^3$.
2. $x^6-6x^4y+12x^2y^2-8y^3$.
3. $a^3+3a+3a^{-1}+a^{-3}$.
4. $x^6y^{-3}-6x^4+12x^2y^3-8y^6$.
5. $\frac{x^3}{8}+\frac{1}{2}+\frac{2}{3x^3}+\frac{8}{27x^6}$.
6. $b^3+\frac{3a^2b^2}{2c^2x^2}+\frac{3a^4b}{4c^4x^4}+\frac{a^6}{8c^6x^6}$.
7. $x^6+6x^5-40x^3+96x-64$.
8. $x^6-6x^5+40x^3-96x-64$.
9. $a^6-9a^4b^3c+27a^2b^6c^2-27b^9c^3$.
10. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.
11. $(a+1)^{6n}x^3-6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x-8c^3a^{3p}$.
12. $8x^6+48x^5y+60x^4y^2-80x^3y^3-90x^2y^4+108xy^5-27y^6$.
13. $x^3-9x^2+30x-45+30x^{-1}-9x^{-2}+x^{-3}$.
14. $a^{3/4}-6a^{5/4}-9a^{3/2}+12a^{7/4}+36a^2+19a^{9/4}-36a^{5/2}-54a^{11/4}$ $-27a^3$.

Find the cube root, preferably by the type-form, of:

15. $\begin{array}{l|l|l|} x^3-6y & x^2+12y^2 & x-\ 8y^3 \\ \quad +9z & \quad -36yz & \quad +36y^2z \\ & \quad +27z^2 & \quad -54yz^2 \\ & & \quad +27z^3 \end{array}$; $\begin{array}{l|l|l|} x^3+6y & x^2+12y^2 & x+\ 8y^3 \\ \quad -9z & \quad -36yz & \quad -36y^2z \\ & \quad +27z^2 & \quad +54yz^2 \\ & & \quad -27z^3 \end{array}$.

16. Arrange the terms in ex. 15 to rising powers of y and then find the cube roots; so, to falling powers of z.

In what are these cube roots alike, and how do they differ?

17. $a^3+6a^2b+12ab^2+8b^3$
$-a^3+6a^2b-12ab^2+8b^3$
$8a^3+12a^2b+6ab^2+b^3$.

18. $x^3+6x^2y+12xy^2+8y^3$
$-x^3+6x^2y-12xy^2+8y^3$
$8x^3+12x^2y+6xy^2+y^3$.

19. In the cube of $a+b$, how many terms are of the form a^3? of the form $3a^2b$? how many terms in all?

Without finding the root, show that $x^3+6x^2y+12xy^2+8y^3$ and $x^3-6x^2y+12xy^2-8y^3$ are perfect cubes.

20. Find the cube of $a+b+c$. How many terms are of the form a^3? of the form $3a^2b$? of the form $6abc$? in all?

Can a polynomial of two terms be a perfect cube? of three terms? of four? of five? of eight? of nine? of ten?

PROB. 10. TO FIND THE ROOT OF A POLYNOMIAL, WHEN THE ROOT-INDEX IS A COMPOSITE NUMBER.

Resolve the root-index into its prime factors;

find that root of the polynomial whose index is the smallest of these factors; [th. 2.

of this root find that root whose index is the next larger factor, and so on.

E.g., the fourth root is the square root of the square root;
the sixth root is the cube root of the square root;
the eighth root is the square root of the square root of the square root.

So,
$$\sqrt[8]{}(x^8-8x^6+28x^4-56x^2+70-56x^{-2}+28x^{-4}-8x^{-6}+x^{-8})$$
$$=\sqrt[4]{}(x^4-4x^2+6-4x^{-2}+x^{-4})$$
$$=\sqrt{}(x^2-2+x^{-2})$$
$$=x-x^{-1}.$$

PROB. 11. TO FIND THE nTH ROOT OF A POLYNOMIAL WHEN n IS PRIME.

Arrange the terms of the polynomial in the order of the powers of some one letter, a perfect nth power first;

write the nth root of the first term as the first term of the root;

divide the second term by n times the $(n-1)$th power of the first term of the root, as a trial divisor, and make the quotient the second term of the root; [th. 1.

from the given polynomial subtract the nth power of the binomial root already found;

divide the first term of the remainder by the same trial divisor as before, and so continue, always subtracting the nth power of the root so far found from the entire polynomial.

E.g.,
$$x^3/27-x^2/3+2x-7+18/x-27/x^2+27/x^3\,(\underline{x/3-1+3/x}$$
$$\underline{x^3/27-x^2/3+\ \ x-1}$$
$$x^2/3)x-6$$
$$x^3/27-x^2/3+2x-7+18/x-27/x^2+27/x^3.$$

QUESTIONS.

Find the fourth root of:

1. $x^8-4x^7+10x^6-16x^5+19x^4-16x^3+10x^2-4x+1$.
2. $x^4+4x^3y+6x^2y^2+4xy^3+y^4+4x^3z+12x^2yz+12xy^2z$ $+4y^3z+6x^2z^2+12xyz^2+6y^2z^2+4xz^3+4yz^3+z^4$.
3. Expand $(a+b)^4$. How many terms of the form a^4? of the form $4a^3b$? of the form $6a^2b^2$? in all?
4. Expand $(a+b+c+d)^4$. How many terms of the form a^4? of the form $4a^3b$? of the form $6a^2b^2$? of the form $12a^2bc$? of the form $24abcd$? in all?
5. Can an expression of four terms be a perfect fourth power? of five terms? of nine? of ten? of fifteen?

Find the sixth root of:

6. $a^6-12a^4+60a^2-160+240a^{-2}-192a^{-4}+64a^{-6}$.
7. $a^6-2a^5b+\frac{5}{3}a^4b^2-\frac{20}{27}a^3b^3+\frac{5}{27}a^2b^4-\frac{2}{81}ab^5+\frac{1}{729}b^6$.
8. $x^{12}-12x^{10}+66x^8-220x^6+495x^4-792x^2+924-792x^{-2}$ $+495x^{-4}-220x^{-6}+66x^{-8}-12x^{-10}+x^{-12}$.

By the process of prob. 11 find the cube root of:

9. $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
10. $x^3-8y^3+27z^3-6x^2y+12xy^2+9x^2z+27xz^2+36y^2z-54yz^2$ $-36xyz$.

Find the fourth root of:

11. $x^4-12x^3+62x^2-180x+321-360x^{-1}+248x^{-2}-96x^{-3}$ $+16x^{-4}$.
12. $(x^2+x^{-2})^2-4(x+x^{-1})^2+12$.

Find the fifth root of:

13. $a^5+10a^4b+40a^3b^2+80a^2b^3+80ab^4+32b^5-5a^4c-40a^3bc$ $-120a^2b^2c-160ab^3c-80b^4c+10a^3c^2+60a^2bc^2+120ab^2c^2$ $+80b^3c^2-10a^2c^3-40abc^3-40b^2c^3+5ac^4+10bc^4-c^5$.

Find the sixth root of:

14. $64x^6-384x^4+960x^2-1280+960x^{-2}-384x^{-4}+64x^{-6}$.

Find these roots correct to three terms:

15. $\sqrt{(1-2x)}$. 16. $\sqrt[3]{(1-3x^2)}$. 17. $\sqrt[4]{(1-4x^3)}$. 18. $\sqrt[5]{(1-5x^4)}$.

§ 5. ROOTS OF NUMERALS.

PROB. 12. TO FIND THE SQUARE ROOT OF A NUMERAL.

Separate the numeral into periods of two figures each, both to the left and to the right of the decimal point;

take the square root of the largest perfect square in the left-hand period;

subtract this square from the period, and to the remainder annex the next period to form the first dividend;

double the root already found and use it as a trial divisor, omitting the last figure of the dividend;

annex the quotient to the root and to the trial divisor;

multiply the complete divisor by this root figure, subtract the product from the dividend, bring down the next period, and proceed as before.

NOTE 1. Numerals are polynomials, but polynomials in which the terms overlie and hide each other; and virtually the rule for finding roots is the same for both.

The separation into periods is a matter of convenience only; it comes from this: that the figures of the root, of different orders, are best found separately, and that, since the square of even tens has two 0's, therefore the first two figures, counting from the decimal point to the left, are of no avail in getting the tens of the root, and are set aside and reserved till wanted in getting the units' figure. So the square of even hundreds has four 0's, and the first four figures, two periods, are set aside and reserved till wanted in getting the tens; and so on.

So, in getting roots of decimal fractions, the square of tenths has two decimal figures, and the first two figures, one period, are used in getting the tenths' figure of the root; the square of hundredths has four decimal figures, and so on. The same thing appears from this: that the root of a fraction is most easily found if the denominator be a perfect square; and this it is, with decimal fractions, only when it consists of 1 with two 0's, or four 0's, or six 0's, and so on; that is, when the number of decimal figures used is even.

QUESTIONS.

Find the square root of:

1. 494209. 2. 3345241. 3. 125457.64.
4. 3533.1136. 5. 17.75358225. 6. 11090466.
7. 1732.323601. 8. 576864324. 9. 1771561.

Find the square root, correct to three decimal places, of:

10. 144. 11. 14.4. 12. 1.44. 13. .144.
14. .0144. 15. .00144. 16. .000144. 17. .0000144.

Find the fourth root, correct to three decimal places, of;

18. 16.0001. 19. 160.001. 20. 1600.01. 21. 16000.1.

Find the square root, correct to a twelfth, of:

22. 49/144. 23. 49/72. 24. 49/36. 25. 49/18.

Find the square root, correct to a ninth, of:

26. 17/9. 27. 17/27. 28. 17/6. 29. 17/36.

30. Find the square of 15; then find $(10+5)^2$, $(9+6)^2$, using the type-form, and show that the powers and products of, 10, 5, 9, 6, overlie and hide each other in the results.

31. In getting the square root of 225, show that 10 is not necessarily the first guess. Make the computation, using 9 for the first number; then, in turn, using 12, 16, 20, for the first number.

32. How many figures in 1^2? in 9^2? in 10^2? in 99^2? in the square of the smallest three-figure number? the largest?

How many figures in the square of a number of n figures?

33. How many figures are there in the square root of an integer of four figures? of three figures? of $2n$ figures? of $2n-1$ figures?

34. What is the object of separating a numeral into two-figure periods? Why could not the process begin with the first figure of the number, irrespective of the decimal point?

35. Why must a decimal fraction that is a perfect square be expressed by an even number of figures?

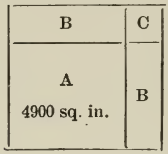
36. How many decimal figures has a perfect nth power?

GEOMETRIC ILLUSTRATION.

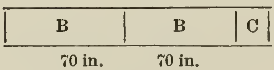
Note 2. The reason of the rule for finding the square root is made clear by a geometric illustration.

E.g., to find the side of a square of 6064 square inches.

The length of the side lies between 70 and 80 inches.



The square A, 70 inches on a side, contains 4900 square inches, leaving 1184 square inches, to be so added that the resulting figure shall be a square. This is done by adding equal rectangles, B, on two sides and a square, C, to complete the figure.



These additions, placed end to end, form a rectangle whose length is a little greater than 140 inches, and whose area is 1184 square inches. The breadth of this rectangle is a little less than the quotient of 1184 by 140. Try 8 inches: then the entire length of B + B + C is 148 inches, and the area (148×8) square inches, is the desired addition.

The side of the square is therefore 78 inches.

CONTRACTION.

Note 3. When the first n figures of a square root have been found by the rule above, then $n-1$ more figures may be got by dividing the remainder by double this root.

In principle, this process is the same as using a trial divisor; only the possible error in the quotient needs consideration.

The whole root is divided into two parts, the first n figures, already found, and the $n-1$ figures that follow.

Put P, Q, for the values of these two parts;
then $\because$ P $\geqslant$ Q $\cdot 10^{n-1}$,

$\therefore 2P$ differs from $2P+Q$, the complete divisor, by less than one part in $2\cdot10^{n-1}$ parts, of this divisor,

and the resulting quotient, Q', differs from the true quotient, Q, by less than one part in $2\cdot10^{n-1}$ parts, of Q.

And $\because Q < 10^{n-1}$,

$\therefore Q'$ is in error by less than half a unit in the last figure.

QUESTIONS.

By contraction find the value, correct to three decimal places, of:

1. $\sqrt{185}$. 2. $\sqrt{912}$. 3. $\sqrt[4]{729}$. 4. $\sqrt[4]{1008}$. 5. $\sqrt[8]{8000}$.

6. If the square root of a number contain three figures and two of them have been found, the part of the root still to be found is less than a tenth of the whole root, and less than a twentieth of the next divisor.

7. The quotient found by using a divisor between twenty twenty-firsts and twenty-one twenty-firsts of the true divisor gives a quotient whose error, if any, is less than a twentieth of the remaining figure. Can this error be half a unit?

8. If three out of five root figures have been found, what part of the whole root is the value of the remaining figures?

The error in the contracted method is less than what fraction of the remainder of the root? what fraction of a unit?

9. If n figures of a root have been found, what is the greatest possible error that can be introduced by finding the next $n-1$ figures by division?

10. If a true quotient be 27365.7 and the division be carried to but five figures, shall the fifth figure written be 5 or 6?

11. If the true root be 27365.7, and if, after finding three figures, the next two be got by division, what danger is there that the fifth figure be written 5 and not 6?

If the true root be 27365.49, what is the danger?

In writing down the last figure of the root, in which direction should an allowance be made?

12. Find the square root of 40297104, and illustrate by a diagram showing the rectangular additions that give the hundreds' figure, the tens' figure, and the units' figure.

PROB. 13. TO FIND THE CUBE ROOT OF A NUMERAL.

Separate the numeral into periods of three figures each, both to the left and to the right of the decimal point;

take the cube root of the largest perfect cube in the left-hand period;

subtract this cube from the period, and to the remainder annex the next period to form the first dividend;

to three times the square of the root already found annex two ciphers and use the result as a trial divisor, placing the quotient as the next figure of the root;

to the trial divisor add three times the product (with one cipher annexed) *of the first part of the root by the new root figure, and add the square of this figure;*

multiply the complete divisor by this figure, subtract the product from the dividend, bring down the next period, and proceed as before.

NOTE 1. The reason for separating the number into periods of three figures is made evident by considering the number of figures in the cubes of numbers having one digit, two digits, three digits, and so on, and the number of zeros in the cubes of exact tens and hundreds. [comp. pr. 12 nt. 1.

CONTRACTION.

NOTE 2. When the first n figures of a cube root have been found as above, then $n-2$ more figures may be got by dividing the remainder by three times the square of this root.

Put P for the value of the figures already found, and Q for that of the $n-2$ figures that follow;

then $\because$ $\text{P} > \text{Q} \cdot 10^{n-1}$,

$\therefore$ 3P^2, the trial divisor, differs from $3\text{P}^2 + 3\text{PQ} + \text{Q}^2$, the complete divisor, by less than one part in 10^{n-1} parts,

and the resulting quotient, Q', differs from the true quotient, Q, by less than one part in 10^{n-1} parts, of Q.

And $\because$ $\text{Q} < 10^{n-2}$,

$\therefore$ Q' is in error by less than a tenth in the last figure.

If the first figure of the root be 2, or larger, then $n-1$ figures may be found by division.

QUESTIONS.

Find the cube root of:

1. 148877.
2. .007821346625.
3. 2439656.927128.
4. 836.802326004904.

Find the cube root, correct to three decimal places, of:

5. 1728.
6. 172.8.
7. 17.28.
8. 1.728.
9. .1728.
10. .01728.
11. .001728.
12. .0001728.

By contraction, find the value correct to three decimal places, of:

13. $\sqrt[3]{625}$.
14. $\sqrt[3]{587}$.
15. $\sqrt[6]{1728}$.
16. $\sqrt[6]{18.625}$.

17. How many figures in 1^3? in 9^3? in 10^3? in 99^3? in the cube of any three-figure number? of any n-figure number?

18. How many figures in the cube root of a number expressed by $3n$ figures? $3n-1$ figures? $3n-2$ figures?

19. If three out of four figures of a cube root have been found; if this part be called a, and the remaining part b; then $3ab+b^2$ is the part of the divisor omitted in the contracted process, b is smaller than a hundredth part of a, $3ab+b^2$ is smaller than a hundredth part of the true divisor, and the error in the quotient is smaller than a tenth part of a unit.

20. If 262144 cubic inches are to be arranged in the form of a cube, and if a cube be formed whose edges are 60 inches, how many cubic inches are so used and how many are left?

If additions 60 inches square made to the top, front, and one side face, would complete a perfect cube, how thick could these additions be?

21. Regarding the integer part of the result in ex. 20 as the second figure of the root, make four other additions, three of them with a length of 60 inches, and a width and thickness each equal to the thickness of the first additions.

What are the dimensions of the final addition required?

How many cubic inches have thus been added?

Draw the original cube, and the several additions.

§ 6. ROOTS OF BINOMIAL SURDS.

THEOR. 6. *If two simple surds of the same degree, in their simplest forms, be equal, their coefficients are equal and their radical parts are equal.*

Let $a\sqrt[n]{A}$, $b\sqrt[n]{B}$ be equal simple surds in their simplest forms;
then will $a=b$, $A=B$.

For, $a:b=\sqrt[n]{(B:A)}$, a true equation, but true only when $\sqrt[n]{(B:A)}$ is rational, [th. 5 cr.

i.e., when $A=B$; and in that case $a=b$ also. Q.E.D.

COR. 1. *Two non-conformable simple surds cannot be equal.*

THEOR. 7. *The sum of two simple non-conformable quadratic surds cannot be rational.*

For, let $\sqrt{A}$, $\sqrt{B}$ be two simple non-conformable quadratic surds, and a, b, c, be rational numbers,

and if possible let $a\sqrt{A}+b\sqrt{B}=c$;

then $\because 2ab\sqrt{AB}=c^2-a^2A-b^2B$, [sqr. both mem. and transp.

and $\sqrt{AB}$, $2ab\sqrt{AB}$, are surds, [pr. 5, nt. 1.

$\therefore$ a surd equals a rational number, which is impossible,

$\therefore a\sqrt{A}+b\sqrt{B}\neq c$. Q.E.D. [df. surd.

COR. 1. *If a, x be rational numbers, $\sqrt{b}$, $\sqrt{y}$ simple quadratic surds, and $x+\sqrt{y}=a+\sqrt{b}$, then $x=a$, $\sqrt{y}=\sqrt{b}$.*

For, if possible, let $x=a+c$, and c be rational;

then $\because a+c+\sqrt{y}=a+\sqrt{b}$, [hyp.

$\therefore \sqrt{b}-\sqrt{y}=c$, which is impossible, [th. 7.

$\therefore x=a$, $\sqrt{y}=\sqrt{b}$.

COR. 2. *If $x+\sqrt{y}=a+\sqrt{b}$, then $x-\sqrt{y}=a-\sqrt{b}$.*

COR. 3. *If $\sqrt{x}+\sqrt{y}=\sqrt{(a+\sqrt{b})}$, then $\sqrt{x}-\sqrt{y}=\sqrt{(a-\sqrt{b})}$.*

For $\because x+y+2\sqrt{xy}=a+\sqrt{b}$, [sqr. both mem.

$\therefore x+y=a$ and $2\sqrt{xy}=\sqrt{b}$, [cr. 1.

$\therefore x+y-2\sqrt{xy}=a-\sqrt{b}$,

and $\sqrt{x}-\sqrt{y}=\sqrt{(a-\sqrt{b})}$. Q.E.D.

Prob. 14. To find a square root of a binomial surd.

Let $a+\sqrt{b}$ be a binomial surd and let $\sqrt{x}+\sqrt{y}\equiv\sqrt{(a+\sqrt{b})}$;
then $\because \sqrt{x}-\sqrt{y}=\sqrt{(a-\sqrt{b})}$, [th. 7 cr. 3.

$$\therefore \sqrt{x}=\tfrac{1}{2}[\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}],$$
$$\sqrt{y}=\tfrac{1}{2}[\sqrt{(a+\sqrt{b})}-\sqrt{(a-\sqrt{b})}],$$
$$x=\tfrac{1}{2}[a+\sqrt{(a^2-b)}], \qquad y=\tfrac{1}{2}[a-\sqrt{(a^2-b)}],$$
$$\sqrt{x}=\sqrt{\tfrac{1}{2}[a+\sqrt{(a^2-b)}]}, \qquad \sqrt{y}=\sqrt{\tfrac{1}{2}[a-\sqrt{(a^2-b)}]},$$
$$\therefore \sqrt{x}+\sqrt{y}=\sqrt{\tfrac{1}{2}[a+\sqrt{(a^2-b)}]}+\sqrt{\tfrac{1}{2}[a-\sqrt{(a^2-b)}]}.$$

QUESTIONS.

1. In the proof of theor. 6, $\sqrt{A}$, $\sqrt{B}$, being surds, $\sqrt{(B/A)}$ can be rational only when $B/A=1$: state the proof.

2. The equation $a\sqrt{A}=b\sqrt{B}$ is satisfied when $a=-b$, $A=B$; but not when $a=b$, $A=-B$, nor when $a=-b$, $A=-B$.

3. If $a+b\sqrt{c}=a'+b'\sqrt{c'}$, then $a=a'$, $b=b'$, $c=c'$.

4. If $\sqrt{a}+\sqrt{b}=\sqrt{c}+\sqrt{d}$, the surds are equal in pairs.

5. A quadratic surd cannot equal the sum of two other non-conformable quadratic surds, nor their difference.

6. If $5-2\sqrt{6}$, $2\sqrt{6}-5$ be the two square roots of $49-20\sqrt{6}$, and $\sqrt{3}-\sqrt{2}$, $\sqrt{2}-\sqrt{3}$, $(\sqrt{3}-\sqrt{2})\sqrt{^-1}$, $(\sqrt{2}-\sqrt{3})\sqrt{^-1}$ be its four fourth roots, what is the relation between the two square roots? between the four fourth roots?

7. If $\sqrt{5}+\sqrt{-3}$ be one of the fourth roots of a binomial surd, what are the others? what are the square roots?

Find a square root of:

8. $7+2\sqrt{10}$.　9. $7+4\sqrt{3}$.　10. $2-\sqrt{3}$.

11. $16-6\sqrt{7}$.　12. $\sqrt{18}-\sqrt{16}$.　13. $9\pm2\sqrt{14}$.

14. $8\sqrt{3}-6\sqrt{5}$.　15. $75-12\sqrt{21}$.　16. $\sqrt{27}+\sqrt{15}$.

17. $-12+6\sqrt{3}$.　18. $2\sqrt{[1+(1-c)^2]}$.　19. $1-2a\sqrt{(1-a^2)}$.

20. $ab+c^2+\sqrt{(b^2-c^2)(a^2-c^2)}$.　21. $xy-2x(xy-x^2)^{1/2}$

22. $9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$.　23. $2(3-\sqrt{2}-\sqrt{3}+\sqrt{6})$.

Find a fourth root of:

24. $28-16\sqrt{3}$.　25. $49+20\sqrt{6}$.　26. $137-36\sqrt{14}$.

27. $a^2+b^2+6ab-4(a^{3/2}b^{1/2}+a^{1/2}b^{3/2})$.　28. $^-8+8\sqrt{^-3}$.

NOTE 1. If a^2-b be a perfect square, so are $a+\sqrt{b}$, $a-\sqrt{b}$.
For $\because \sqrt{x}+\sqrt{y}=\sqrt{(a+\sqrt{b})}$, and $\sqrt{x}-\sqrt{y}=\sqrt{(a-\sqrt{b})}$,

$\therefore x-y=\sqrt{(a^2-b)}$,

and $\sqrt{(a^2-b)}$ is rational if the root sought be possible.

E.g., $7+4\sqrt{3}$ is a perfect square;

for $a=7$, $b=48$, $a^2-b=49-48=1$, a perfect square.

But $7+2\sqrt{3}$ is not a perfect square;

for $a=7$, $b=12$, $a^2-b=49-12=37$, not a perfect square.

NOTE 2. Since $(\sqrt{x}+\sqrt{y})^2=x+y+2\sqrt{xy}$, it appears that the square root of $a+2\sqrt{b}$ is the sum of the square roots of two numbers whose sum is a and whose product is b. This consideration often makes it possible to find such roots by inspection.

E.g., $\sqrt{(-4+\sqrt{-84})}=\sqrt{(-4+2\sqrt{-21})}$;

and $\because -7+3=-4$, $-7\times 3=-21$,

$\therefore \sqrt{(-4+\sqrt{-84})}=\sqrt{-7}+\sqrt{3}$.

So, $\sqrt{(\frac{9}{4}+\sqrt{5})}=\sqrt{(\frac{9}{4}+2\sqrt{\frac{5}{4}})}=1+\frac{1}{2}\sqrt{5}$.

So, $\sqrt{(\sqrt{32}-\sqrt{24})}=\sqrt{(4\sqrt{2}-2\sqrt{6})}=\sqrt{(3\sqrt{2})}-\sqrt{(\sqrt{2})}$
$=\sqrt[4]{18}-\sqrt[4]{2}$.

PROB. 15. TO FIND A CUBE ROOT OF A BINOMIAL SURD.

Let $a+\sqrt{b}$ be a binomial surd, and $x+\sqrt{y}\equiv\sqrt[3]{(a+\sqrt{b})}$; (1)

then $\because x^3+3xy+(3x^2+y)\sqrt{y}=a+\sqrt{b}$, (2) [cube both mem.

$\therefore x^3+3xy-(3x^2+y)\sqrt{y}=a-\sqrt{b}$, (3) [th. 7 cr. 2.

and $x-\sqrt{y}=\sqrt[3]{(a-\sqrt{b})}$, (4)

$\therefore x^2-y=\sqrt[3]{(a^2-b)}$, [mult. eq. 1 by eq. 4.

and the root is possible only if $\sqrt[3]{(a^2-b)}$ be rational.

Put m for $\sqrt[3]{(a^2-b)}$; then $y=x^2-m$,

and $\because x^3+3xy=a$, [add eqs. 2, 3.

$\therefore 4x^3-3mx=a$. [elim. y.

From this point on there is no general solution, but particular examples may be solved by finding a value of x, by inspection, from the equation $4x^3-3mx=a$.

E.g., to find a cube root of $10+6\sqrt{3}$:

then $\because a=10,\quad b=108,\quad m=\sqrt[3]{(100-108)}=-2$;

$\therefore 4x^3+6x=10,$

$\therefore x=1,\quad y=3$; and $1+\sqrt{3}$ is the root sought.

QUESTIONS.

1. Any coefficient may be prefixed to a radical, if the number under the radical sign be divided by that power of the coefficient whose exponent is the index of the radical.

By inspection find a square root of:

2. $8-2\sqrt{7}$. 3. $3\sqrt{5}+\sqrt{40}$. 4. $2\frac{1}{4}-\sqrt{2}$.
5. $x^2+x+2x\sqrt{x}$. 6. $-5-2\sqrt{-6}$. 7. $2a-2\sqrt{(a^2-b^2)}$.
8. $2a-b-2\sqrt{(a^2-ab)}$. 9. $2a+b+2\sqrt{(a^2+ab)}$.

Find a fourth root of:

10. $17+12\sqrt{2}$. 11. $49-20\sqrt{6}$. 12. $14+8\sqrt{3}$.
13. $89+28\sqrt{10}$. 14. $-221-60\sqrt{-1}$. 15. $4a^4$.

Find a cube root of:

16. $7+5\sqrt{2}$. 17. $16+8\sqrt{5}$. 18. $45-29\sqrt{2}$.
19. $22+10\sqrt{7}$. 20. $38+17\sqrt{5}$. 21. $21\sqrt{6}-23\sqrt{5}$.
22. $3a-2a^3+(1+2a^2)\sqrt{(1-a^2)}$. 23. $1+3a+(3+a)\sqrt{a}$.

Of the binomial surds below, which are perfect squares?

24. $4\frac{1}{3}-\sqrt{5\frac{1}{3}}$. 25. $3-\sqrt{2}$. 26. $\sqrt{xy}-\sqrt{(x/y)}$. 27. $9-3\sqrt{5}$.

28. If $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{(a+2\sqrt{b}+2\sqrt{c}+2\sqrt{d})}$, x, y, z must satisfy the four conditions

$$x+y+z=a,\quad xy=b,\quad xz=c,\quad yz=d.$$

Find a square root of $6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$.

29. The square root of $10+2\sqrt{6}+2\sqrt{14}+2\sqrt{21}$ cannot be expressed in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Find a square root of:

30. $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$. 31. $8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}$.
32. $15-2\sqrt{15}-2\sqrt{21}+2\sqrt{35}$. 33. $11+2\sqrt{6}+4\sqrt{3}+6\sqrt{2}$.
34. $15-2\sqrt{3}-2\sqrt{15}+6\sqrt{2}-2\sqrt{6}+2\sqrt{5}-2\sqrt{30}$.

§ 7. QUESTIONS FOR REVIEW.

Define and illustrate:

1. A power; a root; a base; an exponent; a root index.

2. A fraction power; a commensurable power.

3. Powers of a base in the same series; like powers.

4. A radical; a radical factor; a radical expression.

5. A rational radical; a real radical; a surd; an imaginary.

6. A simple radical; a quadratic, a cubic, a quartic, and a quintic radical.

7. Like radicals; conformable radicals.

8. A binomial surd; a trinomial surd.

9. A pair of conjugate quadratic surds; a pair of complementary surds; a group of complementary surds.

State and prove:

10. The binomial theorem.

11. The principle by which a commensurable power of a commensurable power is found.

12. The principle of equal fraction powers.

13. The principle by which the product of two commensurable powers of the same base is found.

How does this principle apply in finding their quotient?

14. The principle by which the product of like commensurable powers of different bases is found.

How does this principle apply in finding their quotient?

15. The principle of the equality of the like parts of two equal simple surds.

16. The principle of the inequality of two non-conformable simple surds.

17. The principle of the equality of the like parts of two equal simple binomial quadratic surds.

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18. What is the product of two conformable simple quadratic surds? their quotient? the product of two such non-conformable surds? their quotient?

Give the general rule, with reasons and illustrations, for:

19. Reducing a simple radical to its simplest form.

20. Freeing a simple radical from coefficients.

21. Adding and subtracting radicals.

22. Multiplying and dividing radicals.

23. Getting powers and roots of radicals.

24. Reducing a fraction with a surd denominator to an equivalent fraction with a rational denominator when the surd denominator is a monomial; when it is a simple quadratic surd; when it is a binomial quadratic surd containing a complex radical; when it is any binomial surd.

25. Rationalizing an equation that contains surds.

26. Finding the square root of a polynomial.

27. Finding the cube root of a polynomial.

28. Finding the root of a polynomial when the root index is a composite number.

29. Finding the nth root of a polynomial when n is prime.

30. Finding the square root of an integer, and of a decimal fraction.

Explain the principle of dividing the numeral into periods; and that of contraction.

31. Finding the cube root of an integer, and of a decimal fraction.

Explain the principle of dividing the numeral into periods; and that of contraction.

32. Finding a root of a fraction.

33. Finding a square root of a binomial quadratic surd.

34. Finding a cube root of a binomial quadratic surd.

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35. If $x^4+6x^3+7x^2-6x+m$ be a perfect square, what is m?

36. If $4x^6+12x^5+5x^4-2x^3+mx^2+nx+p$ be a perfect square, what are the values of m, n, p?

37. Apply the square root process to factoring

$$4x^2+12xy+8y^2+16xz+22yz+15z^2.$$

VII. QUADRATIC EQUATIONS.

An equation that is of the second degree as to its unknown elements is a *quadratic equation.*

E.g., $x^2=9$, $x^2+3x=18$, $ax^2+bx+c=0$. [x unkn.

So, $xy=12$, $ax^2+2hxy+by^2+2gx+2fy+c=0$, [$x$, y, unkn.

So, $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$. [$x$, y, z, unkn.

§ 1. ONE UNKNOWN ELEMENT.

An equation of the form $x^2=9$, is an *incomplete*, or *pure*, *quadratic equation;* one of the form $x^2+3x=18$ is *complete.*

Prob. 1. To solve an incomplete quadratic equation.

Reduce the equation to the type-form $x^2=q$, *and take the square root of each member; then* $x=\pm\sqrt{q}$. [III, ax. 7.

E.g., if $\frac{1}{3}(x^2-10)+\frac{1}{10}(6x^2-100)=3x^2-65$:

then $10x^2-100+18x^2-300=90x^2-1950$, [mult. by 30.

$-62x^2=-1550$, $x^2=25$ and $x=\pm 5$.

There are two square roots, opposites; they are both real if the absolute term be positive, and both imaginary if it be negative. It may seem that the last equation should be $\pm x=\pm 5$; but this gives no new roots.

Prob. 2. To solve a complete quadratic equation.

Reduce the equation to the type-form, $x^2+px+q=0$;

transpose the absolute term, and to each member of the equation add the square of half the coefficient of the first-degree term; [II, pr. 3 nt. 4 frm. 3, 4.

take the square roots of these sums, and solve the two simple equations so found. [III, ax. 7.

The result is of the form $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2-4q)}$.

E.g., if $x^2+3x=40$:

then $x^2+3x+2\frac{1}{4}=42\frac{1}{4}$, [add $(3/2)^2$ to each mem.

$x+1\frac{1}{2}=\pm 6\frac{1}{2}$, [take sqr. rts.

$x=-1\frac{1}{2}\pm 6\frac{1}{2}=5$ or -8.

QUESTIONS.

1. Make a quadratic equation to state that: the area of a square is 4225 square yards; the area of a rectangle is 1200 square rods; the sum of the squares of two numbers is three times their product; the product of the sum and difference of two numbers is 33; the product of two numbers, one 5 less than the other, is 24; the sum of the squares of three numbers increased by twice their products, two by two, is 36.

Which of these equations are complete?

2. How can two independent simple equations be obtained from one quadratic equation? Write the forms for the two roots separately, and find their sum and their product.

Solve the pure quadratic equations:

3. $(x^2+1)(x^2+2)=(x^2+6)(x^2-1)$.

4. $\frac{1}{2}(x^2-\frac{1}{6}a^2)-\frac{1}{3}(x^2-\frac{1}{8}a^2)+\frac{1}{4}(x^2-\frac{1}{10}a^2)=0$.

5. $\frac{1}{6}(3x^2-7)+\frac{1}{9}(25-4x^2)=\frac{1}{3}(5x^2-14)$.

6. $3(5x^2-7)(35-2x)+27(5x^2+7)=9(5x^2-7)(17-\frac{2}{3}x)$.

7. Why is an incomplete quadratic equation called a *pure* quadratic? a complete equation, an *affected* quadratic?

Solve the complete quadratic equations:

8. $x^2-5x+6=0$. 9. $x^2-8x+15=0$. 10. $x^2+10x=-24$.

11. $x^2-5x+4=0$. 12. $6x^2-19x+10=0$.

13. $7x^2-3x=160$. 14. $110x^2-21x+1=0$.

15. $(x-2)^{-1}-2(x+2)^{-1}=3/5$. 16. $\sqrt{(2x+5)}=x+1$.

17. $\dfrac{3x-2}{2x-5}-\dfrac{2x-5}{3x-2}=\dfrac{8}{3}$. 18. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$.

19. $\dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{2x-3}{x-1}$. 20. $\dfrac{x-3}{2x}+\dfrac{x}{x+5}=\dfrac{7}{10}$.

21. $\dfrac{x}{x+3}+\dfrac{x-1}{x-4}=-\dfrac{3}{2}$. 22. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

23. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}$. 24. $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$.

SPECIAL CASES.

NOTE 1. The roots of the equation $x^2+px+q=0$ are
$-\frac{1}{2}p+\frac{1}{2}\sqrt{(p^2-4q)}$, $-\frac{1}{2}p-\frac{1}{2}\sqrt{(p^2-4q)}$, [above.
whose values depend upon the values of p, q.

There are four special cases:

(*a*) *p positive, q negative:*
two real roots, the larger negative, the smaller positive.

(*b*) *p, q both negative:*
two real roots, the larger positive, the smaller negative.

(*c*) *p, q both positive:*
two real roots, both negative, if p^2-4q be positive;
two real roots, both negative and equal to $-\frac{1}{2}p$, if $p^2-4q=0$;
two imaginary roots, conjugates, if p^2-4q be negative.

(*d*) *p negative, q positive:*
two real roots, both positive, if p^2-4q be positive;
two real roots, both positive and equal to $-\frac{1}{2}p$, if $p^2-4q=0$;
two imaginary roots, conjugates, if p^2-4q be negative.

THE SUM AND THE PRODUCT OF THE ROOTS.

NOTE 2. The sum of the two roots is $-p$, and their product is q. A quadratic equation can have not more than two roots.

THE ABSOLUTE TERM ZERO.

NOTE 3. If $q=0$, then of the equation $x^2+px=0$ the two roots are $0, -p$, both real.

SOLUTION BY FACTORING.

NOTE 4. If the expression x^2+px+q be readily factored, then each factor may be put equal to zero, and the two simple equations so found give two values for x.

E.g., to solve the equation $x^2-5x+6=0$:

then $\because x^2-5x+6=(x-2)(x-3)$,

and this product is 0 whether $x-2=0$ or $x-3=0$,

$\therefore$ the roots are 2, 3.

The equation is found by subtracting the roots in turn from x and equating the product of the remainders to 0.

E.g., if the roots be 2, 3, the equation is $(x-2)(x-3)=0$.

QUESTIONS.

1. Whatever be the sign of p, what is that of p^2?

2. If q be negative, what is the sign of p^2-4q? What does this show about the character of the roots? Is $\frac{1}{2}\sqrt{(p^2-4q)}$ then larger or smaller than $\frac{1}{2}p$?

3. If p be positive and q negative, which is the larger, $-\frac{1}{2}p+\frac{1}{2}\sqrt{(p^2-4q)}$ or $-\frac{1}{2}p-\frac{1}{2}\sqrt{(p^2-4q)}$?

4. If q be positive and $p^2=4q$, what are the roots?

5. If $p^2>4q$, are the roots real? if $p^2<4q$?

6. If q be 0, what is the product of the two roots?

7. If p be 0, of what kind is the quadratic?

Write the sum and the product of the roots of the equation:

8. $x^2-4x=60$. 9. $3x^2+6x=24$. 10. $5x^2-15x=140$.
11. $2x^2-3x=12$. 12. $x^2-ax=0$. 13. $5x^2-15x=200$.

14. Suppose the quadratic equation $x^2+px+q=0$ to have three different roots, r, r', r''; what is the value of rr'? of rr''? Prove that two of the supposed three roots are the same.

15. If r, r' be the roots of the equation $x^2-px+q=0$, find the value of $r/r'+r'/r$; of $r^2+r'^2$; of $r^3+r'^3$.

16. Write the expression $x^2-(a+b)x+ab$ in the form $(x-a)(x-b)$: what is the value of this product if $x-a=0$? if $x-b=0$? What are the roots of the equation?

Form a quadratic equation whose roots shall be:

17. 4, -5. 18. $2\frac{1}{2}$, 2. 19. $-\frac{3}{4}$, -8. 20. $a+b$, $a-b$.

By factoring, solve the equations:

21. $y^2+13y=14$. 22. $x^2+7x=30$. 23. $4x^2+12x+9=0$
24. $x^4-a^4=0$. 25. $x^2-5x=14$. 26. $x^3+x^2-x-1=0$.
27. $9x^2-30x+25=0$. 28. $x^3+6x^2-4x=24$.

Put the functions below equal to 0, solve the equations so formed, and by aid of their roots factor the functions:

29. $6x^2-19x+15$. 30. $(x-a)^2-b^2$. 31. $x^2-2mx+m^2-n^2$.
32. $x^2-(m+n)x+(m+p)(n-p)$. 33. x^2+x+1.
34. $x^2-ax-2a^2-b^2+3ab$. 35. x^2-x+1.

GENERAL FORMS.

NOTE 5. The rule for solving complete quadratic equations may be stated in a more general form:

Reduce the equation to the type-form $ax^2+bx+c=0$;

multiply the equation by any number that makes the coefficient of the first term a perfect square;

make the first member a perfect square, take the square root of each member, and solve the simple equations so found.

With a proper multiplier fractions may be avoided.

E.g., if b be even, multiply by a; if odd, by $4a$.

Both rules rest on the form assumed by the square of a binomial, as does that for finding the square root.

The roots are $[-b+\sqrt{(b^2-4ac)}]/2a$, $[-b-\sqrt{(b^2-4ac)}]/2a$.

There are three special cases:

(*a*) *c zero:* then $x=0$, $x=-b/a$, two real roots.

(*b*) *b zero:* then $x=\pm\sqrt{(-c/a)}$,

two real roots, opposites, if a, c be of contrary signs;

two imaginary roots, conjugates, if a, c be of the same sign.

(*c*) *a zero:* then $x=(-b+b)/0$, $x=(-b-b)/0$,

i.e., $x=0/0$, $x=\infty$, an indeterminate and an infinite root.

But this indeterminate root can be determined.

For if $a=0$, and $x\neq\infty$, the equation becomes $bx+c=0$, whose single root is $-c/b$.

This case is specially important as showing the values of x if a be thought of as taking changing values and growing smaller and smaller; for, then, as a grows very small, one of the roots grows very large, and the other approaches $-c/b$.

This is also evident if the equation be written in the form

$$x^{-1}(b+cx^{-1})=-a. \qquad \text{[div. eq. } bx+c=-ax^2 \text{ by } x^2.$$

For, if a grow very small,

then either x^{-1} grows very small and x very large,

or $b+cx^{-1}$ grows very small and x approaches $-c/b$;

and both ∞ and $-c/b$ satisfy the equation when $a=0$.

QUESTIONS.

Solve the equations:

1. $x^2+\frac{7}{3}x=-10/9$. 2. $6x^2+9x=81$. 3. $8x^2-21x+\frac{5}{2}=0$.

4. $x^2-2mx+m^2=n^2$. 5. $6x^2-11x=10$. 6. $\frac{1}{6}x^2-\frac{14}{3}x+30=0$.

7. $a^2x^2+2abx+b^2-c^2=0$. 8. $abx^2+(3b-4a)x=12$.

9. Find the sum and the product of the roots of ax^2+bx+c

If $b=0$, of what kind is the quadratic? what is the sum of the roots? what relation to each other have they?

If c/a be negative, what relation have the roots? if $c=a$?

10. If in ax^2+bx+c, $a=0$, what is one value of x?

Show that the product of the roots is then infinite, and hence find the other root.

11. Discuss the equation $ax^2+bx+c=0$, after the manner of note 1, and show that the two roots are: real and unequal, if $b^2>4ac$; real and equal, if $b^2=4ac$; imaginary, if $b^2<4ac$.

Of the real and unequal roots which is the larger? What conditions make the real roots both positive? both negative?

12. If r, r' be the two roots of ax^2+bx+c, then ax^2+bx+c may be factored and written in the form $a(x-r)(x-r')$.

13. What form has a quadratic equation whose roots are opposites? reciprocals?

14. For what value of c will the equation $2x^2+6x+c=0$ have equal roots? reciprocal roots?

15. If r, r' be the roots of the equation $x^2+px+q=0$, find the equation whose roots are $-r$, $-r'$; $1/r$, $1/r'$.

Without solving, show that the equation

16. $x^2\pm2(p+q)x+2(p^2+q^2)=0$ has imaginary roots.

17. $x^2\pm2(p+q)x+(p+q)^2=0$ has equal roots.

For what value of m will the equation

18. $x^2-15-m(2x-8)=0$ have equal roots?

19. $(x^2-bx)/(ax-c)=(m-1)/(m+1)$, opposite roots?

Without solving, tell the signs of the roots of the equations:

20. $x^2-5x=36$. 21. $x^2+5x=14$. 22. $x^2+\frac{4}{15}x=1/5$.

23. $x^2+5x=-6$. 24. $x^2-5x=-6$. 25. $x^2+5x=-7$.

EQUATIONS SOLVED AS QUADRATICS.

NOTE 6. Equations of the form $ax^{2n}+bx^n+c=0$, or $(ax^{2n}+bx^n+c)^{2m}+p(ax^{2n}+bx^n+c)^m+q=0$, are solved as quadratic equations.

E.g., if $9x^4-52x^2+64=0$:

then $\because 81x^4-468x^2+676=100$, [mult. by 9, add 100.

$\therefore 9x^2-26=\pm 10$, [take sqr. rts.

$\therefore x^2=4$ or $16/9$,

$\therefore x=\pm 2$ or $\pm 4/3$, four real roots.

So, if $(9x^4-52x^2+80)^2+9(9x^4-52x^2+80)-400=0$:

then $\because 4(9x^4-52x^2+80)^2+36(9x^4-52x^2+80)+81=1681$,

$\therefore 2(9x^4-52x^2+80)=-9\pm 41=32$ or -50,

$\therefore 9x^4-52x^2+80=16$ or -25,

and $x=\pm 2$, $\pm 4/3$, $\pm\frac{1}{3}\sqrt{(26\pm\sqrt{-269})}$, eight roots.

So, if $x^4-6x^3+4x^2+15x=14$:

then $(x^2-3x)^2-5(x^2-3x)=14$,

$x^2-3x=5/2\pm\sqrt{(81/4)}$, [solve for x^2-3x.

$x^2-3x=7$ or -2,

$x=3/2\pm\frac{1}{2}\sqrt{37}$ or $3/2\pm 1/2$, [solve for x.

$x=\frac{1}{2}(3\pm\sqrt{37})$, 2, or 1, four roots.

So, if $3x^2-2\sqrt{(3x^2+2x-7)}=10-2x$:

then $3x^2+2x-7-2\sqrt{(3x^2+2x-7)}=3$,

$\sqrt{(3x^2+2x-7)}=1\pm 2=3$ or -1,

$3x^2+2x-7=9$ or 1,

$x^2+\frac{2}{3}x=16/3$ or $8/3$,

$x=-1/3\pm 7/3$ or $-1/3\pm 5/3$,

$x=2$, $-8/3$, or $4/3$, -2, four roots.

QUESTIONS.

Solve the equations:

1. $\sqrt{(2x+7)}+\sqrt{(3x-18)}=\sqrt{(7x+1)}$.

2. $3x+2\sqrt{x}-1=0$.

3. $x^2+3=2\sqrt{(x^2-2x+2)}+2x$.

4. $x^{1/n}-13x^{1/2n}=14$.

5. $\sqrt{(x^2-2x+9)}-\frac{1}{2}x^2=3-x$.

6. $x^4-14x^2+40=0$.

7. $3x^2+15x-2\sqrt{(x^2+5x+1)}=2$.

8. $x^{1/3}+\frac{5}{2}x^{-1/3}=3^{1/4}$.

9. $(x^2+x-6)^2-4(x^2+x-6)=12$.

10. $\sqrt{2x}-7x=-52$.

11. $(x^2+x)^2-3(x^2+x)=108$.

12. $nx^3+x+n+1=0$.

13. $x+5-\sqrt{(x+5)}=6$.

14. $\sqrt[3]{x^2}+3\sqrt[3]{x}=18$.

15. $2x^2+6+3\sqrt{(2x^2+6)}=10$.

16. $x^3(19+x^3)=216$.

17. $(x^2+2)^2+198=29(x^2+2)$.

18. $4=5x^2-x^4$.

19. $\sqrt{(x+4)}-\sqrt{x}=\sqrt{(x+\frac{3}{2})}$.

20. $3x^6+8x^4-8x^2=3$.

21. $(x+1/x)^2-4(x+1/x)=2\frac{1}{4}$.

22. $x^3+x^2-4x-4=0$.

23. $2(\frac{2}{3}x^2-\frac{3}{2})^2+5(\frac{2}{3}x^2-\frac{3}{2})=63$.

24. $x^2+a^2b^2/x^2=a^2+b^2$.

25. $5\sqrt{(3/x)}+7\sqrt{(x/3)}=22\frac{2}{3}$.

26. $x^4+2x^3-3x^2-4x=96$.

27. $x^2+1/x^2+2(x+1/x)=9\frac{1}{4}$.

28. $25/x^2-10/x=3$.

29. $\dfrac{(x-a)^2}{(x-b)(x-c)}+\dfrac{(x-b)^2}{(x-a)(x-c)}+\dfrac{(x-c)^2}{(x-a)(x-b)}=3$.

30. $7/[\sqrt{(x-6)}+4]+12/[\sqrt{(x-6)}+9]+1/[\sqrt{(x-6)}-4]$
$+6/[\sqrt{(x-6)}-9]=0$.

31. $m^2(x+m+17n)(x-m+7n)^2$
$=n^2(x+17m+n)(x+7m-n)^2$.

32. $(x-a+b)^3-(x-a)^3+(x-b)^3-x^3+a^3-(a-b)^3-b^3$
$=(a-b)c^2$.

33. $[x-\sqrt{(x^2-a^2)}]/\sqrt{[x+\sqrt{(x^2-a^2)}]}$
$=\sqrt[4]{(x^2-a^2)}[\sqrt{(x^2+ax)}-\sqrt{(x^2-ax)}]$.

34. $x^2-3x+4+2\sqrt{(x^2-3x+6)}=6$.

35. $ax^{2n}+bx^n+c)^{2m}\pm(ex^n+f)^{2m}=0$.

36. $(ax^{2n}+bx^n+c)^{2m}\pm(dx^{2n}+ex^n)^{2m}=0$.

§ 2. TWO UNKNOWN ELEMENTS.

Prob. 3. To solve a pair of equations involving the same two unknown elements, one equation simple, the other quadratic.

Eliminate one of the unknown elements from the quadratic equation; [III, pr. 2.

solve the resultant for the other unknown element and replace this element by its value in the simple equation;

solve this equation for the first unknown element.

E.g., if $3x+2y=20$, $3x^2+5xy+7y^2=425$:

then $\because x=\frac{1}{3}(20-2y)$, [sol. first eq. for x.

$\therefore \frac{1}{3}(20-2y)^2+\frac{5}{3}y(20-2y)+7y^2=425$, [repl. x in sec. eq.

$\therefore 15y^2+20y=875$, $y=7$ or $-8\frac{1}{3}$, [sol. quad. for y.

$\therefore 3x+2\cdot 7=20$, $x=2$, [repl. y in first eq.

and $3x-2\cdot 8\frac{1}{3}=20$, $x=12\frac{2}{9}$,

and the two pairs of roots are $2, 7$; $12\frac{2}{9}, -8\frac{1}{3}$.

Check. Both pairs of roots satisfy the quadratic equation.

If the two equations be such that they can be combined in one of the familiar forms $x^2\pm 2xy+y^2$, x^2-y^2, the work is shortened by the use of such form.

E.g., if $x+y=13$, $xy=12$:

then $\because x^2+2xy+y^2=169$, [sqr. first eq.

and $4xy=48$, [mult. sec. eq. by 4.

$\therefore x^2-2xy+y^2=121$, $x-y=\pm 11$, [sub. and get sqr. rts.

$\therefore$ from the equations $x+y=13$, $x-y=11$, come $x=12$, $y=1$;

and from the equations $x+y=13$, $x-y=-11$, come $x=1$, $y=12$.

So, if $x^2+y^2=45$, $x-y=-9$:

then $\because x^2-2xy+y^2=81$, [sqr. sec. eq.

$\therefore 2xy=-36$, $x^2+2xy+y^2=9$, $x+y=\pm 3$,

$\therefore x=-6$, $y=3$; $x=-3$, $y=6$.

QUESTIONS.

Find the values of x, y from the pair of equations:

1. $x+y=7,\quad x^2+2y^2=34.$
2. $x-y=12,\quad x^2+y^2=74.$
3. $x+y=a,\quad xy=b^2.$
4. $x-y=a,\quad xy=b^2.$
5. $3x-5y=2,\quad xy=1.$
6. $x+y=100,\quad xy=2400.$
7. $x+y=a,\quad x^2+y^2=b^2.$
8. $x^{1/2}+y^{-1/2}=4,\quad x-y^{-1}=8.$
9. $x+y=4,\ x^{-1}+y^{-1}=1.$
10. $2x+3y=37,\ x^{-1}+y^{-1}=\frac{14}{45}.$
11. $x+y=2,\ x^2-2xy-y^2=1.$
12. $x+y=18,\quad x^3+y^3=4914.$
13. $x+y=72,\quad \sqrt[3]{x}+\sqrt[3]{y}=6.$
14. $x-y=18,\quad x^3-y^3=4914.$
15. $x^2y^{-1}+y^2x^{-1}=9,\ x^{-1}+y^{-1}=3/4.$

16. In the pair of equations $x+y=13,\quad xy=12,$ how are the results affected if x, y exchange places?

Show why either x or y may be 12, and the other be 1.

17. In the pair of equations $x^2+y^2=45,\quad x-y=-9,$ can x, y exchange places? x, $-y$? What relation have x, y?

Solve the pair of equations:

18. $x-y=5,\quad xy=126.$
19. $x+y=8,\quad x^2-y^2=16.$
20. $x-y=4,\quad x^2-y^2=32.$
21. $x+y=11,\quad x^3+y^3=407.$
22. $x-y=4,\ x^3-y^3=988.$
23. $3x-4y=4,\ 9x^2-16y^2=176.$
24. $x^2-y^2=21,\quad x(x+y)^2=45.$
25. $x-y=2,\ x^5-y^5=992.$
26. $3x-2y=10,\quad 3x^2-\frac{1}{2}xy-y^2=80.$
27. $x/y-y/x=3/2,\quad x-y=1.$
28. $x+y=2,\ x^5+y^5=992.$
29. $1/x^3+1/y^3=126/125,\quad 1/x+1/y=6/5.$
30. $x+\frac{2}{3}y=11,\quad x^3+2x^2y+\frac{4}{3}xy^2+\frac{8}{27}y^3=1331.$
31. $5x-y=3,\quad y^2-6x^2=25.$
32. $x-y=2,\quad x^4+y^4=82.$
33. $x+y=1072,\ x^{1/3}+y^{1/3}=6.$
34. $x-y=a,\ x^4+y^4=b^4.$
35. $3x-2y=13,\quad (x+y)^{2/3}+2(x-y)^{2/3}=3(x^2-y^2)^{1/3}.$
36. $7x+5y=29,\quad (2x+y)/(3x-y)-(x-y)/(x+y)=38/15.$
37. $5x-7y=4,\quad (x^3+y^3)/(x+y)^2+(x^3-y^3)/(x-y)^2=43x/8.$
38. $x+y=2,\quad 13(x^5+y^5)=121(x^3+y^3).$
39. $x+y=4,\quad 41(x^5+y^5)=122(x^4+y^4).$
40. $x+y=a,\quad x/(b-y)+(b-y)/x=c.$

Prob. 4. To solve a pair of quadratic equations involving the same two unknown elements.

No one rule is best for all cases; many special devices may be used, and the examples given below suggest methods.

If by combining the old equations, new equations can be found that involve some of the familiar type-forms, such as $x^2 \pm 2xy + y^2$, $x^2 - y^2$, then very often either the square root may be found, or by factoring and division a quadratic equation may be replaced by a simple one.

E.g., if $3xy - 4x - 4y = 0$, $x^2 + y^2 + x + y - 26 = 0$:
put $(x+y)^2 - 2xy$ for $x^2 + y^2$, and write the equations

$$3xy - 4(x+y) = 0, \quad (x+y)^2 + (x+y) - 2xy - 26 = 0,$$

eliminate xy, and solve the quadratic in $(x+y)$;
then $x+y=6$, $xy=8$, or $x+y=-13/3$, $xy=-52/9$;
solve these two pairs of equations for x, y.

So, if $x - y = \frac{1}{4}xy$, $x^2 + y^2 = \frac{5}{2}xy$:
subtract the square of the first equation from the second and solve the resulting equation to find the values of xy;
join each of these equations with the first equation to find values of x, y.

So, if $x^2 + xy + y^2 = 19$, $x^2 - xy + y^2 = 7$:
find the value of xy, then of $x+y$, $x-y$, then of x, y.
There are four pairs of roots.

So, if $\sqrt{(x+y)} + \sqrt{(x-y)} = \sqrt{a}$, $\sqrt{(x^2+y^2)} + \sqrt{(x^2-y^2)} = b$:
then $2x + 2\sqrt{(x^2-y^2)} = a$, $2x^2 + 2\sqrt{(x^4-y^4)} = b^2$, [squaring.
and $x^2 - y^2 = \frac{1}{4}a^2 - ax + x^2$, $x^4 - y^4 = \frac{1}{4}b^4 - b^2x^2 + x^4$.
i.e., $y^2 = ax - \frac{1}{4}a^2$, $y^4 = b^2x^2 - \frac{1}{4}b^4$;

$\therefore b^2x^2 - \frac{1}{4}b^4 = (ax - \frac{1}{4}a^2)^2$, whence x is found; then y.

So, if $(x+y)^2 - 2xy = -(x+y) + 26$, $6xy = 8(x+y)$:
then $(x+y)^2 - \frac{5}{3}(x+y) = 26$; [elim. xy.
and $x+y = 6$ or $-13/3$, $xy = 8$ or $-52/9$;

$x, y = 4, 2;\ 2, 4;\ \frac{1}{6}(-13 \pm \sqrt{377}),\ \frac{1}{6}(-13 \mp \sqrt{377})$.

QUESTIONS.

Solve the pair of equations:

1. $4x^2+7y^2=148,\ 3x^2-y^2=11.$ 2. $x+y=x^2,\ 3y-x=y^2.$

3. $x^2+y^2=\frac{5}{2}xy,\quad x-y=\frac{1}{4}xy.$ 4. $x^2+xy=6,\ x^2+y^2=5.$

5. $x^2+y^2=a^2,\quad xy=b^2.$ 6. $x^3+y^3=9,\quad xy=2.$

7. $x^3+xy^2=10,\quad y^3+x^2y=5.$ 8. $x^2+y=4x,\ y^2+x=4y.$

9. $x^2(x+y)=80,\quad x^2(2x-3y)=80.$

10. $x^4+x^2y^2+y^4=133,\quad x^2-xy+y^2=7.$

11. $x^2=ax+by,\quad y^2=ay+bx.$

12. $x+y=10,\quad \sqrt{xy^{-1}}+\sqrt{yx^{-1}}=5/2.$

13. $bx+ay=ab,\quad bx+ay=4xy.$

14. $8x^{1/3}-y^{1/2}=14,\quad x^{2/3}y^{3/2}=2y^2.$

15. $81x^4-16y^4=1296,\quad 9x^2+4y^2=36.$

16. $x^2+xy=63,\ y^2+xy=18.$ 17. $x-y=9,\ x^3-y^3=243.$

18. $x^2-xy+y^2=25,\ x^3+y^3=125.$ 19. $3(x+y)=15,\ xy=6.$

20. $4x+4y=12,\quad x^3+y^3=63.$ 21. $x-y=3,\quad xy=4.$

22. $x^3+y^3=3(x+y),\quad x-y=1.$ 23. $x^2+y^2=13,\ -xy=6.$

24. $x^2+y^2=25,\quad xy-x+y=-5.$

25. $4(x+y)=3xy,\quad x+y+x^2+y^2=26.$

26. $x+y+\sqrt{xy}=14,\quad x^2+y^2+xy=84.$

27. $xy+6x+7y=50,\quad 3xy+2x+5y=72.$

28. $x^4+x^2y^2+y^4=243,\quad x^2-xy+y^2=27.$

29. $x^4-x^3y+x^2y^2-xy^3+y^4=31,\quad x^5+y^5=31.$

30. $x^2+y^2+x+y=4,\quad 2xy+3x+3y=8.$

31. $x-y-2\sqrt{(x-y)}=-1,\quad x^3-y^3+4\sqrt{(x^3-y^3)}=60.$

32. $x+y=5/6,\quad x^2y^2+1/x^2y^2+4(xy+1/xy)=60\frac{25}{36}.$

33. $8x/y=50y/x,\quad xy+x-y=13.$

34. $x^2/y^2+(2x+y)/\sqrt{y}=20-(y^2+x)/y,\quad x+8=4y.$

CHANGE OF THE UNKNOWN ELEMENTS.

Sometimes the solution of a pair of equations may be simplified by changing the unknown elements.

E.g., if $x+y=4,\quad x^4+y^4=82$:

put $u+v$ for x, $u-v$ for y;

then $(u+v)+(u-v)=4,\quad u=2,$

and $(u+v)^4+(u-v)^4=82,\quad u^4+6u^2v^2+v^4=41,$

$v^4+24v^2-25=0,$

$v^2=1$ or $-25,\quad v=\pm 1$ or $\pm 5\sqrt{^-1},$

$x=3,\quad 1,\quad 2+5\sqrt{^-1},\quad 2-5\sqrt{^-1},$

$y=1,\quad 3,\quad 2-5\sqrt{^-1},\quad 2+5\sqrt{^-1}.$

So, if $x^4-x^2+y^4-y^2=84,\quad x^2+x^2y^2+y^2=49$:

then $(x^2+y^2)^2-2x^2y^2-(x^2+y^2)=84,\quad (x^2+y^2)+x^2y^2=49;$

put u for x^2+y^2, v for x^2y^2, and solve.

There are eight pairs of roots.

BOTH EQUATIONS OF THE FORM $ax^2+bxy+cy^2=k$.

If both equations be of the form $ax^2+bxy+cy^2=k$, then:

replace y by vx, eliminate x^2 by comparison, and solve the resultant for v;

replace v by its values in either of the vx-equations and solve for x;

get the products of corresponding values of v, x for values of y.

Check. *Replace x, y by their pairs of values in the other original equation, and see whether it be satisfied thereby.*

E.g., if $2xy+5y^2=195,\quad 3x^2-4xy=7$:

then $\because 2vx^2+5v^2x^2=195,\quad 3x^2-4vx^2=7,$ [repl. y by vx.

$\therefore 7(2v+5v^2)=195(3-4v)$ and $v=5/7$ or $-117/5;$

$\therefore 3x^2-\frac{20}{7}x^2=7,\quad x=\pm 7,\quad y=\pm 5,$ if $v=5/7.$

and $3x^2+\frac{468}{5}x^2=7,\quad x=\pm\sqrt{(5/69)},\quad y=\mp 117/\sqrt{345}$ if $v=-117/5$, four pairs of roots.

Check. $2\cdot 7\cdot 5+5\cdot 5^2=195,\quad 2\cdot{}^-7\cdot{}^-5+5\cdot({}^-5)^2=195;$ and so for the other pair of roots.

QUESTIONS.

Solve the pair of equations:

1. $x+y=5,\quad x^4+y^4=97.$ 2. $x^2+3xy=54,\ xy+4y^2=115.$
3. $x-y=3,\quad x^5-y^5=3093.$ 4. $x^{1/2}+y^{1/4}=1,\ x^2+y=17.$
5. $x^2+xy+4y^2=6,\quad 3x^2+8y^2=14.$
6. $x^4+y^4=14x^2y^2,\quad x+y=9.$ 7. $x^2-y^2=a^2,\quad xy=b^2.$
8. $x^2+y^2=45,\ x+y+\sqrt{2xy}=15.$
9. $x^2+y^2=7+xy,\quad x^3+y^3=6xy-1.$
10. $x^2-xy\sqrt{2}+y^2=2,\quad x^4+y^4=20.$
11. $xy(x+y)=30,\quad x^3+y^3=35.$
12. $x^2y+xy^2=48,\quad x^2y-xy^2=16.$
13. $x+y+1=0,\quad x^5+y^5+211=0.$
14. $x^2+y^2+xy=15\frac{1}{4},\quad x^2-y^2=2\frac{1}{4}.$
15. $1/x^{1/3}-1/y^{1/3}=1,\quad 1/x-1/y=37.$
16. $x^2/y+y^2/x=9/2,\quad 3/(x+y)=1.$
17. $x^2+xy+2y^2=74,\quad 2x^2+2xy+y^2=73.$
18. $x^4-x^2+y^4-y^2=84,\quad x^2+x^2y^2+y^2=49.$
19. $x^3+y^3+3x+3y=378,\quad x^3+y^3-3x-3y=324.$
20. $x^2+y^2+x+y=50,\quad xy+x+y=29.$
21. $x^4-x^2y^2+y^4=16,\quad 2x^4+3x^2y^2-3y^4=32.$
22. $x^2+a^2=y^2+b^2=(x+y)^2+(a-b)^2.$
23. $x^2y+xy^2=30,\quad x^4y^2+x^2y^4=468.$
24. $x^2+xy+y^2=84,\quad x-\sqrt{xy}+y=6.$
25. $x^4-y^4=240,\quad (x+y)^2=36.$
26. $x^my^n=(3/2)^{m-n},\quad x^ny^m=(2/3)^{m-n}.$
27. $\sqrt{(x/y)}+\sqrt{(y/x)}=10/3,\quad x+y=10.$
28. $x^2/y^2+y/x+x/y=27/4-y^2/x^2,\quad x-y=2.$
29. $(x+y)(x^3-y^3)=819,\quad (x-y)(x^3+y^3)=399.$
30. $x^4+x^2y^2+y^4=931,\quad x^2-xy+y^2=19.$

§ 3. THREE OR MORE UNKNOWN ELEMENTS.

PROB. 5. TO SOLVE A SYSTEM OF n QUADRATIC EQUATIONS, INVOLVING THE SAME n UNKNOWN ELEMENTS.

The examples given below suggest methods.

E.g., if $x(x+y+z)=18$, $y(x+y+z)=12$, $z(x+y+z)=6$:

then $(x+y+z)(x+y+z)=36$, [add.

$x+y+z=\pm 6$, [take sqr. rt.

and $x=\pm 3$, $y=\pm 2$, $z=\pm 1$. [div. eqs. 1, 2, 3 by $x+y+z$.

So, if $xyz=a^2(y+z)=b^2(z+x)=c^2(x+y)$: divide by xyz;

then $1=a^2(1/zx+1/xy)=b^2(1/xy+1/yz)=c^2(1/yz+1/zx)$.

and $1/yz=\frac{1}{2}(-1/a^2+1/b^2+1/c^2)$,

$1/zx=\frac{1}{2}(1/a^2-1/b^2+1/c^2)$,

$1/xy=\frac{1}{2}(1/a^2+1/b^2-1/c^2)$;

and $\because x^2=1/yz:(1/zx\cdot 1/xy)$, [identity.

$\therefore x^2=2(-1/a^2+1/b^2+1/c^2)$

$:[1/a^2-1/b^2+1/c^2)\cdot(1/a^2+1/b^2-1/c^2)]$;

and so for y^2, z^2.

So, if $y^2+yz+z^2=7$, $z^2+zx+x^2=13$, $x^2+xy+y^2=3$:

then $(x-y)(x+y+z)=6$, [sub. eq. 1 from eq. 2.

$(z-y)(x+y+z)=10$. [sub. eq. 3 from eq. 2.

$\therefore x-y:z-y=3:5$, $5x-5y=3z-3y$, $x=\frac{1}{5}(2y+3z)$;

$\therefore z^2+\frac{1}{5}(2y+3z)\cdot\frac{1}{5}(2y+8z)=13$;

[repl. x by $\frac{1}{5}(2y+3z)$ in eq. 2.

combine this equation with eq. 1, and solve for y, z;

then $y=\pm 2$, $\pm 1/\sqrt{19}$; $z=\mp 3$, $\pm 11/\sqrt{19}$;

and $x=\mp 1$, $\pm 7/\sqrt{19}$. [repl. y in eq. 3.

So, if $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$:

from the square of each equation subtract the product of the other two;

then $\because x(x^3+y^3+z^3-3xyz)=a^2-bc$,

$y(x^3+y^3+z^3-3xyz)=b^2-ca$,

and $\quad z(x^3+y^3+z^3-3xyz)=c^2-ab.$

$\therefore x/(a^2-bc)=y/(b^2-ca)=z/(c^2-ab),$

$\therefore x=\pm(a^2-bc)/\sqrt{(a^3+b^3+c^3-3abc)}.$ [repl. y, z in eq. 1.

So, $\quad y=\pm(b^2-ca)/\sqrt{(a^3+b^3+c^3-3abc)},$

$z=\pm(c^2-ab)/\sqrt{(a^3+b^3+c^3-3abc)}.$

QUESTIONS.

Find the values of x, y, z from the set of equations:

1. $yz=bc,\quad bx+ay=ab,\quad cx+az=ac.$

2. $xyz=10,\quad yz/x=10,\quad xy/z=10.$

3. $x+y+z=3\frac{1}{2},\quad x^{-1}+y^{-1}+z^{-1}=3\frac{1}{2},\quad xyz=1.$

4. $9x+y-8z=0,\quad 4x-8y+7z=0,\quad yz+zx+xy=47.$

5. $xy=a(x+y),\quad xz=b(x+z),\quad yz=c(y+z).$

6. $x+y+z=6,\quad 4x+y-2z=0,\quad x^2+y^2+z^2=14.$

7. $x+y+z=13,\quad x^2+y^2+z^2=65,\quad xy=10.$

8. $x+2y-z=11,\quad x^2-4y^2+z^2=37,\quad xz=24.$

9. $3x+y-2z=0,\quad 4x-y-3z=0,\quad x^3+y^3+z^3=467.$

10. $x+y+z=a,\quad x^2+y^2+z^2=a^2,\quad x^3+y^3+z^3=a^3,$

11. $x+y=z,\quad 3x-2y+17z=0,\quad x^3+3y^3+2z^3=167.$

12. $(x+y)(y+z)=30,\quad (x+z)(y+x)=15,\quad (z+x)(z+y)=18.$

13. $(y-z)(z+x)=22,\quad (z+x)(x-y)=33,\quad (x-y)(y-z)=6.$

14. $x^2y^2z^2w=12,\quad x^2y^2zw^2=8,\quad x^2yz^2w^2=1,\quad xy^2z^2w^2=4/3.$

15. $(x+v)(y+z)=-a+b+c,\quad (v+y)(x+z)=a-b+c,\quad (v+z)(x+y)=a+b+c,\quad x^2+y^2+z^2+v^2=3(a+b+c).$

16. $xyz/(x+y)=-8,\quad xyz/(y+z)=24,\quad xyz/(x+z)=12.$

17. $xy^2z^3=108,\quad yz^2/x=18,\quad yx^2/z=2/3.$

18. $x^3y^2z=12,\quad x^2yz^3=54,\quad x^7y^3z^2=72.$

19. $3x^2+2y^2+5z^2=0,\quad 7x^2\ \ 3y^2-15z^2=0,\quad 5x-4y+7z=6.$

§ 4. QUESTIONS FOR REVIEW.

Define and illustrate:

1. A complete quadratic equation.

By what other name is such an equation known?

2. An incomplete, or pure, quadratic equation.

Give the general rule, with reasons and illustrations, for:

3. Solving an incomplete quadratic equation.

4. Solving a complete quadratic equation of the form $x^2+px+q=0$; of the form $ax^2+bx+c=0$.

5. Solving a pair of quadratic equations involving the same two unknown elements, when one equation is simple and the other quadratic; when both equations are quadratic.

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6. Discuss the equation $x^2+px+q=0$: what are the roots if p be positive and q be negative? if p, q be both negative? if p, q be both positive? if p be negative and q be positive?

Find the sum of the roots, and their product.

Show what relation between p and q makes the two roots equals; opposites; reciprocals; opposite reciprocals.

7. Discuss the equation $ax^2+bx+c=0$: what are the roots if c be zero? if b be zero? if a be zero?

What relation have a, b, c, if the two roots be real and equal? if real and unequal? if imaginary?

Can one root be real and the other imaginary?

8. Show how to form a quadratic equation that shall have two given numbers for its roots.

9. If α, β be the roots of the equation $ax^2+bx+c=0$, find the value, in terms of a, b, c, of $\alpha-\beta$, $\alpha^2+\beta^2$, $\alpha^2-\beta^2$, $\alpha^3+\beta^3$, $\alpha^3-\beta^3$, $\alpha/\beta+\beta/\alpha$, $\alpha^2/\beta+\beta^2/\alpha$.

10. So, find the equation whose roots are:

$$\alpha/\beta,\ \ \beta/\alpha;\quad \alpha/\beta,\ \ -\beta/\alpha;\quad \alpha^2,\ \ \beta^2;\quad 1/\alpha^2,\ \ 1/\beta^2.$$

11. Show how to solve a quadratic equation by factoring.

12. Show how to factor a quadratic function by solving the quadratic equation formed by putting the function equal to 0.

13. Show how a pair of equations that involve the same two unknown elements may sometimes be solved by changing the unknown elements.

14. Solve the equation $2x^4-9x^3+14x^2-9x+2=0$.
[Divide by x^2; then $2(x^2+x^{-2})-9(x+x^{-1})+14=0$;
write y for $x+x^{-1}$; then $2(y^2-2)-9y+14=0$;
solve this equation for y, then the equation $x+x^{-1}=y$ for x.

Such equations, in which terms equidistant from the ends of the function have equal coefficients, are *reciprocal equations*, and their roots come in *reciprocal pairs*.

THE ROOTS OF $^+1$ AND OF $^-1$.

15. The three cube roots of $^+1$ are 1, $\frac{1}{2}(-1\pm\sqrt{-3})$.
[Write $x=\sqrt[3]{1}$; then $x^3-1=0=(x-1)(x^2+x+1)$;
put these two factors, in turn, equal to 0, and solve the equations so formed for x.

16. The sum of the three cube roots of $^+1$ is 0, and so is the sum of their products in pairs; the product of the last two roots is $^+1$, and each of them is the square of the other.

Express the three cube roots of 1 by 1, r, r^2, and show that:

17. $1+r+r^2=0$. 18. $(1-r+r^2)(1+r-r^2)=4$.

19. $(1+r^2)^2=r^2$. 20. $(1-r)\cdot(1-r^2)\cdot(1-r^4)\cdot(1-r^5)=9$.

21. $(1-r+r^2)\cdot(1-r^2+r^4)\cdot(1-r^4+r^8)\cdots 2n$ factors $=2^{2n}$.

22. Find the three cube roots of $^-1$.

The sum of these three roots is 0, and so is the sum of their products in pairs; and either of the last two roots is the opposite of the square of the other.

23. What are the cube roots of $^+a^3$ and of $^-a^3$?

24. Find the four fourth roots of $^+1$.

The sum of these four roots is 0, and so are the sums of their products in pairs and in threes; and the product of the four roots is $^-1$.

25. The four fourth roots of $^-1$ are $(1+i)\cdot\sqrt{\frac{1}{2}}$, $(1-i)\cdot\sqrt{\frac{1}{2}}$, $(1+i)\cdot-\sqrt{\frac{1}{2}}$, $(1-i)\cdot-\sqrt{\frac{1}{2}}$. [$i\equiv\sqrt{^-1}$.

26. What are the fourth roots of $^+a^4$ and of $^-a^4$?

27. The five fifth roots of $^{+}1$ are

$$1,\quad \tfrac{1}{4}[-1\pm\sqrt{5}\pm i\sqrt{(10+2\sqrt{5})}].$$

[Write $x^5-1=0=(x-1)\cdot(x^4+x^3+x^2+x+1)$;
then $x-1=0$ and $x=1$:
and $\therefore x^2+x+1+x^{-1}+x^{-2}=0$; [put sec. fact. $=0$, div. by x^2.

$\therefore x^2+2+x^{-2}+x+x^{-1}=1,\quad (x+x^{-1})^2+(x+x^{-1})+\frac{1}{4}=\frac{5}{4}$,

$\therefore x+x^{-1}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$; and these two quadratic equations, when solved, give the last four values above.

The sum of these five roots is 0, and so are the sums of their products in pairs, in threes, and in fours; and the product of the five roots is $^{-}1$.

28. Find the five fifth roots of $^{-}1$.

29. What are the fifth roots of $^{+}a^5$ and of $^{-}a^5$?

30. The six sixth roots of $^{+}1$ are the three cube roots of $^{+}1$ and the three cube roots of $^{-}1$.

Group these six roots in three pairs of opposites.

31. The six sixth roots of $^{-}1$ are $\pm i$, $\pm\frac{1}{2}(\sqrt{3}\pm i)$.

[Write $x^6+1=0=(x^2+1)(x^4-x^2+1)$; put these two factors, in turn, equal to 0, and solve the equations so formed.]

32. What are the sixth roots of $^{+}a^6$ and of $^{-}a^6$?

MAXIMA AND MINIMA.

33. If x vary, but remain real, what is the least possible value of x^2-4x+3?

[Write $x^2-4x+3=y$; then $(x-2)^2=y+1$,
and $\because (x-2)^2$ is always positive,

$\therefore y+1$ is positive and $^{-}1$ is the value sought.

34. Find the least value possible of x^2+px+q, and find the value of x which makes this function least.

35. If x vary, but remain real, what bounds has the fraction $(x^2+2x-11)/(x-3)$? what are the like values of x?

[Write $(x^2+2x-11)/(x-3)=y$; then $x^2+(2-y)x+3y-11=0$,
and $\because x$ can be real only when $(2-y)^2 \nless 12y-44$,

i.e., when $y^2-16y+48 \not< 0$ and $(y-4)(y-12)$ is positive,

$\therefore$ the fraction has no value between 4 and 12, but has all other values.

Equate the fraction to 4 and 12, in turn, and solve for x.

36. The fraction $(x+a)/(x^2+bx+c^2)$ lies always between two fixed bounds if $a^2+c^2>ab$ and $b^2<4c^2$; it lies always beyond two fixed bounds if $a^2+c^2>ab$ and $b^2>4c^2$; and it takes all values if $a^2+c^2<ab$.

In the light of ex. 36, find the bounds of these fractions:

37. $(x+4)/(x^2+2x+8)$. 38. $(x+2)/(x^2+8x+4)$.

39. $(x+8)/(x^2+4x+2)$. 40. $(x+4)/(x^2+8x+2)$.

41. $(x+2)/(x^2+4x+8)$. 42. $(x+8)/(x^2+2x+4)$.

43. Divide a given number a into two parts, (a) whose product shall be a maximum, (b) the sum of whose squares shall be a minimum.

44. Of all the rectangular fields that have the same area, the square has the shortest perimeter, and the shortest diagonal. [Use the equations

$$(x+y)^2=(x-y)^2+4xy, \quad x^2+y^2=(x-y)^2+2xy.$$

45. Of all the rectangles that have the same perimeter, the square has the greatest area. Which of them has the least ?

46. Find the condition that the quadratic function

$$ax^2+2hxy+by^2+2gx+2fy+c$$

may be resolved into two linear factors.

[Put the function equal to 0 and solve the equation for x; then $ax+hy+g=\pm\sqrt{[y^2(h^2-ab)+2y(hg-af)+(g^2-ac)]}$; and this radical is rational only if $(hg-af)^2=(h^2-ab)(g^2-ac)$, *i.e.*, if $abc+2fgh-af^2-bg^2-ch^2=0$.

In the light of ex. 46, show whether these functions can be factored, and, if possible, factor them.

47. $5x^2+12xy-9y^2-24x-18y-5$.

48. $5x^2+12xy-9y^2-24x+18y+5$.

49. $5x^2-18xy+9y^2+24x-12y-5$.

50. $5x^2-18xy+9y^2+24x+18y+5$.

VIII. THE THREE PROGRESSIONS, INCOMMENSURABLE POWERS, AND LOGARITHMS.

A series is an expression such that each term is connected with the preceding terms by some law. [comp. df. series, p. 34.

§ 1. ARITHMETIC PROGRESSION.

A series is in *arithmetic progression* if each term after the first be found by adding a constant to the term before it. The constant added is the *common difference*. A series in arithmetic progression is an *arithmetic series*.

The abbreviations are: a first term, l last term, d common difference, n number of terms, s sum of all the terms.

If d be positive, the series is a *rising series;* if d be negative, it is a *falling series*.

E.g., 1, 3, 5, 7, 9 is a rising series,
wherein $d={}^{+}2$, $a=1$, $l=9$, $n=5$, $s=25$.

So, 9, 7, 5, 3, 1, ${}^{-}1$, ${}^{-}3$, is a falling series,
wherein $d={}^{-}2$, $a=9$, $l={}^{-}3$, $n=7$, $s=21$.

THEOR. 1. *In an arithmetic series,* $l=a+(n-1)d$.

For $\because$ a, $a+d$, $a+2d$, $a+3d, \cdots$, $a+(k-1)d$ are the first, second, third, fourth, $\cdots k$th terms, [df.

$\therefore$ $a+(n-1)d=l$, the last of a series of n terms. Q.E.D.

THEOR. 2. *In an arithmetic series,* $s=\frac{1}{2}n(a+l)$.

For $\because$ $s=a+(a+d)+(a+2d)+\cdots+(l-d)+l$, n terms,

and $s=l+(l-d)+(l-2d)+\cdots+(a+d)+a$, n terms,

$$\therefore 2s=(a+l)+(a+l)+(a+l)+\cdots+(a+l), \quad n \text{ times},$$

$$=n\cdot(a+l),$$

$$\therefore \quad s=\tfrac{1}{2}n(a+l).$$ Q.E.D.

NOTE. The numbers a, d, l, n, s are the *five elements* of an arithmetic series. In theors. 1, 2 these elements are connected by two equations; hence, any three of them being given, the other two can be found, and in all there are twenty equations: four that contain no a, four that contain no d, and so on.

QUESTIONS.

1. Write an arithmetic series in which $a=3$, $d=^-2$, $n=5$.

2. What two conditions would make negative all the terms of an arithmetic series? Can an arithmetic series be an endless rising series and have some terms negative? all negative?

3. From the equation $l=a+(n-1)d$, find the value of a in terms of l, n, d.

4. So, solve this equation for n, and for d.

5. Solve the equation $s=\frac{1}{2}n(a+l)$, in turn for n, a, l.

6. In the series of integers $1, 2, 3, \cdots$, find the last term and the sum of 5 terms; of 20 terms; of n terms; of $2n$ terms; of $2n+1$ terms.

7. In the series of odd positive integers, find the sum of 35 terms; of 50 terms; of n terms; of $2n$ terms; of $2n+1$ terms.

8. In the series of even positive integers, find the sum of n terms; of $2n$ terms; of $2n-1$ terms; of $2n+1$ terms.

9. Find the five elements of the arithmetic series

$$1, 3, 5, \cdots 99; \quad 1, 3, 5, \cdots 2k-1; \quad 4+5+\cdots=5350.$$

10. In an arithmetic series, the sum of two terms equidistant from the extremes is constant.

11. What multiple of the common difference must be added to the fifth term of an arithmetic series to give the ninth? to the twelfth to give the twentieth?

12. If $a=1$, $d=2$, then $s=n^2$.

13. A three digit number is 26 times the sum of its digits, which are in arithmetic progression; if 396 be added to the number the order of the digits is reversed: find the number.

14. A hundred stones are laid in a row, a meter apart, and a basket is placed a meter from the first stone: how many kilometers will a man walk who, starting from the basket, picks up all the stones, one by one, and brings them separately to the basket?

15. The sum of three numbers in arithmetic progression is 27 and the sum of their squares is 293: find the numbers.

ARITHMETIC MEANS—INTERPOLATION.

PROB. 1. TO INTERPOLATE m ARITHMETIC MEANS BETWEEN TWO NUMBERS a, l.

Divide the remainder, $l-a$ *by* $m+1$ *for the common difference; to* a *add one, two, three $\cdots$ times this difference.*

E.g., to interpolate 5 means between 12 and 48:

then $\because (48-12):(5+1)=6$, the common difference,

$\therefore$ the series sought is 12, 18, 24, 30, 36, 42, 48.

NOTE. By aid of this problem, from every arithmetic series a new arithmetic series may be formed by interpolating the same number of arithmetic means between every pair of consecutive terms; and the common difference of this new series is the quotient of the common difference of the other divided by one more than the number of terms so interpolated.

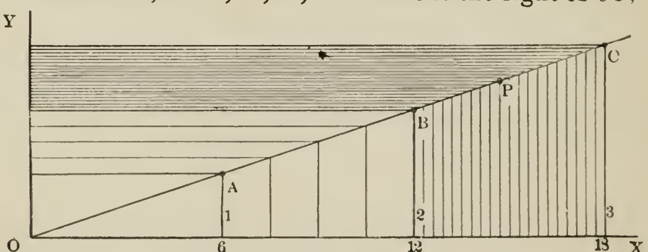
E.g., if two means be interpolated between pairs of consecutive terms,

then the series 6, 12, 18, 24,

becomes the series 6, 8, 10, 12, 14, 16, 18, 20, 22, 24.

GEOMETRIC ILLUSTRATION.

Let OX, OY be two straight lines at right angles to each other; take points A, B, C, $\cdots$ such that they are 1, 2, 3, $\cdots$ inches above OX, and 6, 12, 18, $\cdots$ inches to the right of OY;



then A, B, C, $\cdots$ lie on a straight line through O, and their distances from OX are in arithmetic progression, and so are their distances from OY, and from O;

and the common differences of these three series are 1, 6, $\sqrt{37}$.

Between A, B, C, $\cdots$ interpolate other equidistant points;
then three new arithmetic series are formed, however close together the points of division may lie.

CONTINUOUS PROGRESSION.

Lay a pencil on the figure and move it slowly to the right, keeping it always parallel to OY and letting it cut AB at a moving point P;
then three series are formed, in *continuous arithmetic progression:* the growing distances of P from OX, from OY, and from O.

QUESTIONS.

Find the five elements of the arithmetic series:

1. 5, 6 means, 75.
2. 3, 6 means, -11.
3. $2\frac{1}{2}$, 4 means, 20.
4. 5 terms, 19, 7 means, 67.

5. When m arithmetic means are interpolated between two given terms, these two terms and the means make a series of how many terms? What are a and l in this new series?

6. Form a new arithmetic series by interpolating three means between every pair of terms of the series 4, 8, 12, $\cdots$ What is the new common difference?

Form a new series by taking every fifth term of the series last formed, beginning with the second.

What is now the common difference?

7. Show that, at simple interest, the given principal and its amounts for successive years form an arithmetic series, wherein n is one more than the number of years. What are a, d, l?

Write such a series for five years, and show that the final amount agrees with the formula $l = a + (n-1)d$.

8. If the interest be reckoned half-yearly, how many means must be interpolated between every two terms? if quarterly? if monthly? What is d in each of these new series?

If the interest be reckoned instantly, what kind of a series results? what is the number of terms? the common difference?

What two elements of the series are unchanged?

§ 2. GEOMETRIC PROGRESSION.

A series is in *geometric progression* if each term after the first be found by multiplying the term before it by a constant multiplier. This multiplier is the *common ratio.* A series in geometric progression is a *geometric series.*

The abbreviations are: a first term, l last term, r common ratio, n number of terms, s sum of all the terms.

If r be larger than 1, the series is a *rising series;* if smaller, a *falling series.*

E.g., 1, 2, 4, 8, 16 is a rising series,
wherein $r={}^+2$, $a=1$, $l=16$, $n=5$, $s=31$.

So, 1, $^-2$, 4, $^-8$, 16 is a rising series,
wherein $r={}^-2$, $a=1$, $l=16$, $n=5$, $s=11$.

But 16, 8, 4, 2, 1, 1/2, 1/4 is a falling series,
wherein $r=1/2$, $a=16$, $l=1/4$, $n=7$, $s=31\frac{3}{4}$.

THEOR. 3. *In a geometric series,* $l=ar^{n-1}$.

For $\because a, ar, ar^2, \cdots ar^{k-1}$ are the first, second, third, $\cdots$ kth terms, [df.

$\therefore ar^{n-1}=l$, the last of a series of n terms. Q.E.D.

THEOR. 4. *In a geometric series,* $s=(a-rl)/(1-r)$.

For $\because s \;=a \;+ar \;+ar^2+\cdots+lr^{-2}+lr^{-1}+l$, [df.

$\therefore rs=ar+ar^2+ar^3+\cdots+lr^{-1}+l \quad +lr$,

$\therefore s-rs=a-lr$, and $s=(a-rl)/(1-r)$. Q.E.D.

COR. 1. *In an infinite falling geometric series, l is indefinitely small; and the value of s is indefinitely near to the quotient* $a/(1-r)$.

COR. 2. *A repeating decimal equals a common fraction whose numerator consists of the repeating figures, and the denominator of as many 9's as there are repeating figures.*

E.g., the decimal $.\dot{4}\dot{5}$, *i.e.*, .454545 $\cdots$, is the sum of the
geometric series $45/100+45/100^2+45/100^3+\cdots$

and $s=45/100/(1-\frac{1}{100})=45/99$.

QUESTIONS.

1. If in a geometric series the first term be positive and the ratio a positive proper fraction, what signs have the terms? how do they change? if the ratio be a negative proper fraction?

2. Solve the equation $l=ar^{n-1}$ for a and for r.

3. In a geometric series $a={}^-3$, $l={}^-48$, $r={}^-2$: what is n?

4. Solve the equation $s=(a-rl)/(1-r)$ in turn for a, r, l.

5. Find the last term and the sum of 10 terms of the series of integer powers of ${}^\pm2$; of n terms; of $2n$ terms.

Find the 12th term and the sum of 12 terms of the series:

6. $1- \quad + \quad -3/4+\cdots$ 7. $2/5+3/5^2+2/5^3+3/5^4+\cdots$

8. If r be a proper fraction, how do rising powers of r vary? What is the value of a very high power of such a ratio?

Find the value of:

9. $.212121\cdots$ 10. $.\dot{6}7\dot{2}$. 11. $.3\dot{6}8\hat{4}$. 12. $.152727\cdots$

13. The geometric mean between two positive numbers lies between them, and is their mean proportional.

14. By what power of the common ratio must the fifth term of a geometric series be multiplied to give the ninth term? the twelfth term to give the twentieth term?

15. In a geometric series the product of any two terms equidistant from a given term is the square of that term.

State and prove the like truth about an arithmetic series.

16. If all the terms of a geometric series be multiplied (or divided) by the same number, the products (or quotients) form a geometric progression with the same ratio as before.

17. From the two fundamental equations $l=ar^{n-1}$, $s=(a-rl)/(1-r)$, eliminate a, l, r, in turn.

18. A man invests \$100 in stocks that pay 3 per cent half-yearly dividends, and invests the dividends, as received, at the same rate: how much has he invested at the end of 5 years?

19. The sum of three numbers in geometric progression is 13, and the product of the mean by the sum of the extremes is 30; what are the numbers?

GEOMETRIC MEANS.

PROB. 2. TO INTERPOLATE m GEOMETRIC MEANS BETWEEN TWO NUMBERS, a, l.

Take the $(m+1)$th root of the quotient l/a for the common ratio; multiply a by the first, second, $\cdots$ powers of this ratio.

E.g., to interpolate three means between 3 and 48:

then $\because \sqrt[4]{(48:3)}=2$, the common ratio,

$\therefore$ the series sought is 3, 6, 12, 24, 48.

NOTE. By aid of this problem, from every geometric series a new geometric series may be formed by interpolating the same number of geometric means between every pair of consecutive terms; and the common ratio of this new series is that root of the common ratio of the other whose index is one more than the number of means so interpolated.

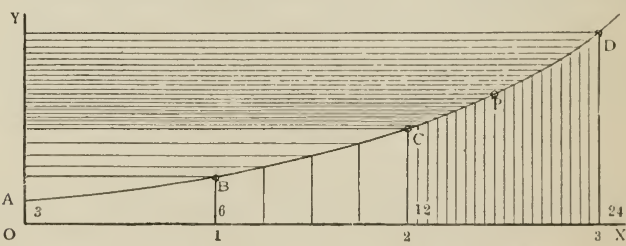
E.g., if two means be interpolated between pairs of consecutive terms,

then the series 3, 6, $12\cdots$

becomes the series 3, $3\sqrt[3]{2}$, $3\sqrt[3]{4}$, 6, $6\sqrt[3]{2}$, $6\sqrt[3]{4}$, $12\cdots$

GEOMETRIC ILLUSTRATION.

Let OX, OY be two straight lines at right angles to each other; take points A, B, C, $\cdots$ such that they are 0, 1, 2, $\cdots$ inches to the right of OY and 3, 6, 12, $\cdots$ inches above OX;



then A, B, C, $\cdots$ lie on a curve; their distances from OY are in arithmetic progression, with a common difference 1, but their distances from OX are in geometric progression, with a common ratio 2.

Between A, B, C, $\cdots$ interpolate other points whose distances from OY are arithmetic means between the terms of the series $1, 2, 3, \cdots$

and whose distances from OX are the like geometric means between the terms of the series $3, 6, 12, \cdots$

CONTINUOUS PROGRESSION.

Lay a pencil on the figure and move it slowly to the right, keeping it always parallel to OY, and letting it cut the curve at a moving point P;

then the growing distance of P from OY forms a series in continuous arithmetic progression,

and the growing distance of P from OX forms a series in *continuous geometric progression.*

QUESTIONS.

1. From the formula $r = \sqrt[n-1]{(l/a)}$ find the new ratio when m geometric means are interpolated between every two terms.

Insert geometric means:

2. Four between 1 and 32. 3. Two between 1 and 1000.

4. Three between 1/9 and 9. 5. Three between 2 and 1/8.

6. Form a new geometric series by interpolating three terms between each pair of terms of the series $3, 9, 27, \cdots$

What is the new common ratio?

Form a new series by taking every fifth term of this series, beginning with the second. What is now the common ratio?

7. In compound interest the principal and its amounts at the ends of successive years form a geometric series.

Show that $a = p(1 + \text{rate})^t$ agrees with the formula $l = ar^{n-1}$.

8. If the interest be compounded half-yearly, but in such a way as not to change the final amount, how many means are inserted between every two terms? if quarterly? if monthly?

What is r in each of these new series?

How must the interest be compounded to make the amount a continuous variable? What is then the value of n? of r?

What two elements of the series are unchanged?

§ 3. HARMONIC PROGRESSION.

A series is in *harmonic progression* if the reciprocals of the terms be in arithmetic progression.

E.g., $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$; $3, 4, 6, 12, \cdots$

The last term of a harmonic series is found by computing the last term of the arithmetic series of reciprocals and inverting it; the sum of a harmonic series can be found only by the actual addition of the terms.

THEOR. 5. *Of three consecutive terms of a harmonic series the first is to the third as the excess of the first over the second is to the excess of the second over the third.*

Let p, q, r be three consecutive terms of a harmonic series;

then $p : r = p - q : q - r$.

For $\because 1/q - 1/p = 1/r - 1/q$, [df.

$\therefore (p - q)/pq = (q - r)/qr$

and $(p - q)/(q - r) = pq/rq = p/r$. Q.E.D.

PROB. 3. TO INTERPOLATE m HARMONIC MEANS BETWEEN TWO NUMBERS, a, l.

Find m arithmetic means between the reciprocals of a, l, and take the reciprocals of these means.

E.g., to interpolate two harmonic means between 12 and 48;

then $\because 1/12 - 1/48 = 3/48$, and $3/48 : 3 = 1/48$,

$\therefore$ the arithmetic series is $1/12,\ 1/16,\ 1/24,\ 1/48$,

and the harmonic series is $12,\ 16,\ 24,\ 48$.

THE ANALOGIES OF THE THREE PROGRESSIONS.

NOTE. The analogies and relations of the three progressions may be thus stated: if p, q, r be three numbers

in arithmetic progression, then $p - q : q - r = p : p$;
in geometric progression, then $p - q : q - r = p : q$;
in harmonic progression, then $p - q : q - r = p : r$.

The three means are $\frac{1}{2}(p + r)$, $\sqrt{pr}$, $2pr/(p + r)$; and the geometric mean of p, r is the geometric mean of their arithmetic and harmonic means.

QUESTIONS.

Continue the harmonic series for three terms in each direction:

1. 2, 3, 6. 2. 3, 4, 6. 3. 1, $1\frac{1}{6}$, $1\frac{2}{5}$. 4. $1\frac{1}{7}$, $1\frac{1}{3}$, $1\frac{3}{5}$.

5. The harmonic mean of two numbers is twice the product of the numbers divided by their sum.

Insert harmonic means as follows:

6. Five between 1/3 and 1/5.

7. Three between 7/5 and 7/13.

8. Five between 4/5 and $-8/11$.

9. Three between $1/(4a+b)$ and $1/5b$.

10. Whatever be the values of p, r, $(p-r)^2$ is positive, and $p^2+r^2>2pr$; $\frac{1}{2}(p+r)>\sqrt{pr}$, $4p^2r^2/(p+r)^2<pr$.

11. Prove the analogies of the three progressions as above.

12. The geometric mean between two numbers is 8, the harmonic mean $6\frac{2}{5}$: find the numbers.

13. The difference of two numbers is 8 and their harmonic mean is $1\frac{4}{5}$: what are the numbers?

14. The arithmetic, geometric, and harmonic means of two numbers greater than unity are in descending order of magnitude.

15. The arithmetic mean between two numbers is 3 and the harmonic mean $2\frac{2}{3}$: find the numbers.

16. If z be the harmonic mean of a, b, then
$$1/(z-a)+1/(z-b)=1/a+1/b.$$

17. What number must be added to each of three given numbers a, b, c, that the three results may be in harmonic progression?

18. If a, b, c be in harmonic progression, then
$$1/(a-b)+1/(b-c)+4/(c-a)=1/c-1/a.$$

19. If to each of three consecutive terms of a geometric progression the second of the three be added, the sums are in harmonic progression.

20. If G be the geometric mean of A, B, then the harmonic means of A, G, and G, B are equal.

§ 4. INCOMMENSURABLE POWERS.

If $r^1, r^2, r^3, \cdots r^n$ be a series in geometric progression, the exponents $1, 2, 3, \cdots n$ form a series in arithmetic progression; and the numbers $1, 2, 3, \cdots n$ serve also to define the position of the terms of the geometric series, r^1 being the first term, T_1, r^2 the second term, T_2, r^3 the third term, $\text{T}_3, \cdots r^n$ the nth term, T_n.

If two geometric means be interpolated between every pair of consecutive terms of this geometric series, the exponents of r in the new series form the arithmetic series $1, 1\frac{1}{3}, 1\frac{2}{3}, 2, 2\frac{1}{3}, 2\frac{2}{3}, 3, \cdots$; and here, too, the exponents serve to define the position of the terms of the geometric series, the terms of the original series being the *major terms,* or *integer terms,* and the others the *minor terms,* or *fraction terms.*

So, for other means that may be interpolated between pairs of terms, however great their number; and r^n is the nth term, T_n, whether n be an integer or a fraction, positive or negative. The exponents form an arithmetic series of commensurable numbers, and the corresponding terms of the geometric series are commensurable powers of r.

But if the arithmetic series be made continuous, then also the geometric series is a continuous series of the powers of r. Among the terms of this continuous arithmetic series are included incommensurable numbers, and the corresponding terms of the continuous geometric series are incommensurable powers of r, *incommensurable powers* being distinguished from commensurable powers as powers whose exponents are incommensurable numbers.

LEMMA. *The ratio of two terms of a geometric series is that power of the common ratio whose exponent is the excess of the number which defines the position of the first term over that which defines the other.* [df. geom. prog.

E.g., $\text{T}_9 : \text{T}_5 = r^4$, $\text{T}_{20} : \text{T}_{12} = r^8$, $\text{T}_{1/2} : \text{T}_{1/3} = r^{1/6}$, $\text{T}_p : \text{T}_q = r^{p-q}$,

$\text{T}_{2\sqrt{2}} : \text{T}_{\sqrt{2}} = r^{\sqrt{2}}$, $\text{T}_{\sqrt{3}} : \text{T}_{\sqrt{2}} = r^{\sqrt{3}-\sqrt{2}}$, $\text{T}_{m\pi} : \text{T}_{n\pi} = r^{(m-n)\pi}$.

QUESTIONS.

1. In the geometric series $\cdots$ 1/64, 1/8, 1, 8, 64, $\cdots$ find the common ratio. What is T_0? T_1? T_{-1}? T_2? T_{-2}? T_3? T_{-3}?

Interpolate two means between every pair of consecutive terms, and name the terms so interpolated.

2. In the geometric series $\cdots$, 16/81, 4/9, 1, 9/4, 81/16, $\cdots$, find the common ratio, and name the terms.

Interpolate three means between every pair of consecutive terms, and name the terms so interpolated.

3. On a horizontal line as axis take a point O, and on this axis lay off distances to the right and left from O proportional to the positive and negative exponents of the series in ex. 2; at the points so found draw vertical lines and lay off distances upward proportional to the terms themselves; join the upper ends of these vertical lines, in their order, by straight lines, thus forming a *plat* of the series, that rises faster and faster.

4. If a dollar be put at compound interest at the annual rate of 10 per cent, find the amount at the end of 1 year; 2 years; 3 years; $\cdots$ 8 years. These amounts form a true geometric series with the common ratio 1.1.

5. In ex. 4, the amount would be the same at the end of any period of years if the interest were compounded half yearly at the ratio $\sqrt{1.1}$, quarterly at the ratio $\sqrt[4]{1.1}$, monthly at the ratio $\sqrt[12]{1.1}$, and so for any shorter periods.

What relations have the arithmetic series of exponents, the growing time, and the geometric series of amounts?

6. In ex. 4, as the time grows continuously, so may the interest and the amount, *i.e.,* just as soon as any interest is earned, that interest may itself become principal and begin to earn interest; and the plat of the growing amount is a continuous curve, rising faster and faster from the axis.

7. Growing continuously, the amount in ex. 4 becomes double the principal at some time between seven and eight years. This time is definite and distinct; it is not an integer, and not a simple fraction; hence it is incommensurable, and 2 is an incommensurable power of 1.1.

THE PRODUCT OF POWERS OF THE SAME BASE.

THEOR. 6. *The product of two or more powers of a base is that power of the base whose exponent is the sum of the exponents of the factors.*

Let m, n be any two numbers, A any base, then $A^m \cdot A^n = A^{m+n}$.

(*a*, *b*) *m, n both commensurable:* [VI, th. 4.

(*c*) *m, n either or both of them incommensurable.*

For, let the geometric series $A^1, A^2, A^3, \cdots$, whose common ratio is A, be made continuous by letting the exponent grow continuously through all the intermediate values, commensurable and incommensurable;

then $\because A^m = T_m$, whatever m may be, and $A^{m+n} = T_{m+n}$, [df.

and $\because T_{m+n}$ is the nth term beyond T_m, and its value is $A^m \cdot A^n$, [lem.

$\therefore A^m \cdot A^n = A^{m+n}$;

and so for three or more powers. Q.E.D.

A POWER OF A POWER.

THEOR. 7. *A power of a power of a base is that power of the base whose exponent is the product of the given exponents.*

Let m, n be any two numbers, A any base; then $(A^m)^n = A^{m \cdot n}$.

(*a*, *b*) *m, n both commensurable:* [VI, th. 2.

(*c*) *m, n either or both of them incommensurable.*

For, let the geometric series $A^1, A^2, A^3, \cdots$, whose common ratio is A, be made continuous, and, in this series, mark $A^m, (A^m)^2, (A^m)^3, \cdots$ as the principal terms of a series whose common ratio is A^m;

then $\because (A^m)^n$ is both the nth term of this series and the mnth term of the original series, whose value is $A^{m\,n}$, [df.

$\therefore (A^m)^n = A^{m\,n}$. Q.E.D.

THE PRODUCT OF LIKE POWERS OF DIFFERENT BASES.

THEOR. 8. *The product of like powers of two or more bases is the like power of their product.*

Let n be any number and $A, B, C, \cdots$ any bases;

then $A^n \cdot B^n \cdot C^n \cdots = (A \cdot B \cdot C \cdots)^n$.

(*a, b*) *n commensurable:* [VI, th. 5.

(*c*) *n incommensurable.*

For, let $B = A^m$;

then $A^n \cdot B^n = A^n \cdot A^{m \cdot n}$ [th. 7.

$= A^{n+m \cdot n}$ [th. 6.

$= A^{(1+m)\cdot n} = (A \cdot A^m)^n$ [th. 7.

$= (A \cdot B)^n$; [above.

and so for three or more bases. Q.E.D.

QUESTIONS.

1. If the interest be compounded instantly at a rate that is equivalent to the annual rate 10, and if the principal be doubled in m years, and this double be tripled in n years more, the whole time is $m+n$ years, and the final amount is sixfold the first principal:

i.e., if $2p = p \cdot (1.1)^m$, and $6p = 2p \cdot (1.1)^n$,

then $\because 6p = p \cdot (1.1)^m \cdot (1.1)^n$, and $6p = p \cdot (1.1)^{m+n}$,

$\therefore (1.1)^m \cdot (1.1)^n = (1.1)^{m+n}$.

2. If the interest be compounded instantly at a rate that is equivalent to the annual rate 10, and if the principal be doubled in m years, and this double be tripled in n periods of m years, the whole time is $m \cdot n$ years, and the final amount is sixfold the first principal:

i.e., if $2p = p \cdot (1.1)^m$ and $6p = p \cdot 2^n$;

then $\because 6p = p \cdot [(1.1)^m]^n$, and $6p = p \cdot (1.1)^{m \cdot n}$.

$\therefore [(1.1)^m]^n = (1.1)^{m \cdot n}$.

§ 5. LOGARITHMS.

The *logarithm* of a number is the exponent of that power to which another number, the *base*, must be raised to give the number first named.

E.g., in the equation $A^x = N$, A is the base, N is the number, and x is the exponent of A and the logarithm of N.

Operations upon or with logarithms are therefore operations upon or with the exponents of the powers of the same base; and the principles established for such powers apply directly to logarithms, with but the change of name noted above.

The three equations $A^x = N$, $x = \log_A N$, $N = \log_A^{-1} x$ are equivalent equations. The second is read, *x is the logarithm of* N *to the base* A, or *x is the* A*-logarithm of* N; and the last means that N is the number whose logarithm to the base A is x; it is read, N *is the anti-logarithm of x to the base* A.

E.g., $0 = \log_A 1$ and $1 = \log_A^{-1} 0$, whatever A may be.

So, $1 = \log_2 2$, $2 = \log_3 9$, $3 = \log_4 64$, $4 = \log_5 625$,

and $2 = \log_2^{-1} 1$, $9 = \log_3^{-1} 2$, $64 = \log_4^{-1} 3$, $625 = \log_5^{-1} 4$,

So, $^-1 = \log_2 1/2$, $^-2 = \log_3 1/9$, $^-3 = \log_4 1/64$, $^-4 = \log_5 1/625$,

and $^-1 = \log_{1/2} 2$, $^-2 = \log_{1/3} 9$, $^-3 = \log_{1/4} 64$, $^-4 = \log_{1/5} 625$.

If the base be well known it may be suppressed, and these two equations may then be written $x = \log N$, $N = \log^{-1} x$.

If while A is constant N take in succession all possible values from 0 to ∞, the corresponding values of x constitute *a system of logarithms to the base* A.

THEOR. 9. *The logarithm of unity to any base is zero.*

THEOR. 10. *The logarithm of the base itself is unity.*

THEOR. 11. *If the base be positive and larger than unity, the logarithms of numbers greater than unity are positive, while of numbers positive and smaller than unity they are negative; and if the base be positive and smaller than unity, the logarithms of numbers greater than unity are negative, while of numbers positive and smaller than unity they are positive.*

QUESTIONS.

With 4 as base, find the logarithms of:

1. 16; 8; 1; 64; 32; 256; 4^{-1}; 16^{-3}; $32^{-3/4}$.

2. .25; $16\sqrt{2}$; 1/16; 1/8; 128/1024; $\sqrt{2}/256$.

3. Are the numbers below commensurable or incommensurable powers of 4?

5; 25; 125; 625; .5; .25; .125; .0625.

Between what commensurable powers do the incommensurable powers lie?

4. What is the logarithm of 144 to the base $2\sqrt{3}$?

5. What effect is produced on the logarithm of a number by making the base smaller? by making it larger?

6. Find $\log_5 3125$; $\log_7 343^{-1}$; $\log_{1/3} 81$; $\log_{1/7} 343$.

7. Find, to the base a, $\log \sqrt[3]{a^{-15/2}}$; $\log [(a^{-5/3})^{-3/7}]^{-7/9}$.

8. With 9 as base find the anti-logarithm of:

$\frac{1}{2}$; $^-1$; 5/2; $^-3/2$; 2; $^-2$; 0; 1; 3/4; $^-3/4$.

9. With 8 as base, write a series of six logarithms.

What base makes:

10. $\log 64 = 2$? $\log 6\frac{1}{4} = 2$? $\log^{-1} 3 = -1000$? $\log 125 = 3$?

11. $\log^{-1} \frac{3}{2} = 343$? $\log^{-1} - \frac{5}{2} = 32$? $\log 2\frac{1}{4} = \frac{1}{2}$? $\log 64 = {}^-3$?

12. A number has different logarithms to different bases, and it may have the same logarithm to two different bases, but with a given base, it can have but one logarithm.

13. With a negative base what positive numbers and what negative numbers have logarithms? What negative numbers have logarithms to positive bases?

In making computations by logarithms, when may the signs of the numbers be disregarded?

14. With a base smaller than unity, what is the logarithm of a very large number? of unity? of a very small number? of zero? of the base itself?

15. With a positive base larger than unity, what is the logarithm of a very small fraction? of zero? of a very large number? of unity? of the base itself?

LOGARITHMS OF PRODUCTS AND QUOTIENTS.

THEOR. 12. *The logarithm of a product is the sum of the logarithms of the factors; and the logarithm of a quotient is the excess of the logarithm of the dividend over that of the divisor.* [th. 6.

E.g., $\log_A (B \cdot C : D) = \log_A B + \log_A C - \log_A D$.

LOGARITHMS OF POWERS AND ROOTS.

THEOR. 13. *The logarithm of a power of a number is the product of the logarithm of the number by the exponent of the power sought; that of a root is the quotient of the logarithm by the root-index.* [th. 7.

E.g., $\log_A (B^2 \cdot \sqrt[3]{C}) = 2 \log_A B + \frac{1}{3} \log_A C$.

CHANGE OF BASE.

THEOR. 14. *If the logarithm of a number be taken to two different bases, the first logarithm is the product of the second logarithm into the logarithm of the second base taken to the first base, and vice versa.*

For, let N be any number, A, B two bases, and $y = \log_B N$;

then $\because$ $N = B^y$, [df. log.

$\therefore \log_A N = \log_A B^y = y \cdot \log_A B$, [th. 13.

$= \log_B N \cdot \log_A B$. Q. E. D.

So, $\log_B N = \log_A N \cdot \log_B A$. Q. E. D.

COR. 1. *The logarithms of two numbers, each taken to the other number as base, are reciprocals.*

For, let $B = A^x$,

then $\because$ $A = B^{1/x}$, $\log_A B = x$, $\log_B A = 1/x$, [df. log.

$\therefore \log_A B \cdot \log_B A = x \cdot 1/x = 1$. Q. E. D.

NOTE. There are two systems of logarithms in use: *natural logarithms*, whose base is e [$2.71828\cdots$], and *common logarithms*, whose base is 10. Their relations to each other are expressed by the equations

$$\log_{10} N = \log_e N \cdot \log_{10} e, \quad \log_{10} e = .434294.$$

QUESTIONS.

If the logarithms of x, y, a, b be known, show how to find:

1. $\log(\sqrt{x^2}\cdot\sqrt{y^3})$. 2. $\log(\sqrt{a^{-2}b}\cdot\sqrt[3]{ab^{-3}})$. 3. $\log abxy$.

4. $\log(ab:xy)^2$. 5. $\log(ax:by)^{-1/2}$. 6. $\log(a^{1/2}b^{-1/3}x^{-1/4}:y^{1/5})$.

Given $\log_{10}2=.3010$, $\log_{10}3=.4771$, $\log_{10}7=.8451$, find:

7. $\log 5$.	8. $\log 6$.	9. $\log 8$.	10. $\log 9$.
11. $\log 10$.	12. $\log 12$.	13. $\log 14$.	14. $\log 15$.
15. $\log 16$.	16. $\log 20$.	17. $\log 18$.	18. $\log 21$.
19. $\log 24$.	20. $\log 25$.	21. $\log 28$.	22. $\log 30$.
23. $\log\sqrt{72}$.	24. $\log 300^{1/2}$.	25. $\log 16^3$.	26. $\log 1728$.
27. $\log 2/5$.	28. $\log 3\frac{1}{2}$.	29. $\log 4\frac{1}{2}$.	30. $\log 12\frac{1}{4}$.
31. $\log 1\frac{2}{3}$.	32. $\log\frac{1}{3}\sqrt{15}$.	33. $\log\sqrt[3]{(3^2\cdot 5^4:\sqrt{2})}$.	

34. $\log\sqrt[4]{(729\cdot\sqrt[3]{9^{-1}}\cdot 27^{-4/3})}$. 35. $\log(27^{-1/3}:64^{-1/6})$.

36. What is the logarithm of the arithmetic mean of 15, 21? of the harmonic mean? of the geometric mean?

37. The logarithms of two given numbers bear a constant ratio to each other, whatever the base.

38. Given $\log_7 49=2$, and $\log_{10}7=.8451$, find $\log_{10}49$.

39. Given $\log_{16}64=3/2$, $\log_{10}16=1.2040$, find $\log_{10}64$.

40. If $\log x^2y^3=a$, $\log x/a=b$, find $\log x$ and $\log y$.

From the logarithms of 2, 3, 5, 7 to the base 10, above, find:

41. $\log_7 10$.	42. $\log_5 10$.	43. $\log_3 10$.	44. $\log_2 10$.
45. $\log_7 700$.	46. $\log_5 7$.	47. $\log_3\sqrt{30}$.	48. $\log_3 1\frac{1}{2}$.
49. $\log_5 2$.	50. $\log_2 800$.	51. $\log_3 3\frac{1}{3}$.	52. $\log_3 270$.
53. $\log_2 5$.	54. $\log_5 343$.	55. $\log_5 28$.	56. $\log_7 14\frac{2}{7}$.

57. $\log 75/16-2\log 5/9+\log 32/243=\log 2$.

From $\log_{10}2=.30103$ and $\log_{10}7=.84509$, find:

58. $\log_7\sqrt{2}$. 59. $\log_{\sqrt{2}}7$. 60. $\log_2\sqrt{7}$. 61. $\log_{\sqrt{7}}2$.

SPECIAL PROPERTIES OF THE BASE 10.

The logarithm of an integer power of 10 is an integer.

E.g., of 1000, 100, 10, 1, .1, .01, .001,

the logarithms to the base 10 are

$^{+}3, \quad ^{+}2, \quad ^{+}1, \quad 0, \quad ^{-}1, \quad ^{-}2, \quad ^{-}3.$

But of any other number the logarithm is fractional or incommensurable; and if incommensurable, it consists of a whole number, the *characteristic*, and an endless decimal, the *mantissa*.

As a matter of convenience the mantissa is always taken positive; and the characteristic is the exponent, positive or negative, of that integer power of 10 which lies next below the given number.

A negative characteristic is indicated by the sign $-$ above it.

E.g., the logarithms of the numbers

	2000,	20,	.2,	.002,
are	$3.30103\cdots$,	$1.30103\cdots$,	$\bar{1}.30103\cdots$,	$\bar{3}.30103\cdots$,

whose characteristics are 3, 1, $\bar{1}$, $\bar{3}$, and whose common mantissa is $.30103\cdots$.

THEOR. 15. *If a number be multiplied (or divided) by any integer power of* 10, *the logarithm of the product (or quotient) and the logarithm of the number have the same mantissa.*

For $\because$ the logarithm of a product is the sum of the logarithms of its factors, [th. 12.

and $\because$ the logarithm of the multiplier is an integer, [hyp.

$\therefore$ the mantissa of the sum is identical with the mantissa of the logarithm of the multiplicand. Q.E.D.

So, if a number be divided by an integer power of 10.

COR. *The logarithms of all numbers expressed by the same figures in the same order have the same mantissa, but different characteristics.*

E.g., the logarithms, correct to four figures, of the numbers

	79500,	795,	7.95,	.0795,	.000795,
are	4.9004,	2.9004,	0.9004,	$\bar{2}.9004$,	$\bar{4}.9004$.

QUESTIONS.

1. What kind of power of 10 is a number whose logarithm is a simple fraction?

Can such a power of 10 be a commensurable number?

Conversely, what kind of logarithm has a commensurable number that is not an integer power of 10?

What is the mantissa of such a logarithm?

2. What kind of series is formed by the powers of 10 on the opposite page? what by their logarithms?

So, with the series 2000, 20, .2, .002 and their logarithms?

3. Name some base that gives a rising series of logarithms for a falling series of numbers.

4. What relation has the difference in the series of logarithms to the ratio in the corresponding series of numbers?

5. Explain why the logarithms of 9520, 95.2, .952, .00952, have the same mantissa.

What are the characteristics of these logarithms?

6. Moving the decimal point one place to the left in a number has what effect on the characteristic of its logarithm?

So, moving the point three places to the right? ·

Given $\log 409600\dot{0} = 6.6124$, find:

7. log 4096. 8. log 40.96. 9. log 6.4. 10. log 8.

11. log 4. 12. log 512. 13. log .016. 14. log .0002.

15. How many figures in 10^2? in any number between 10^2 and 10^3? in a number between 10^{n-1} and 10^n?

What is the characteristic of the logarithm of an integer expressed by three figures? by n figures? How many figures are there in the anti-logarithm if the characteristic be 5?

16. Given $\log 2 = .30103$, how many figures are there in 2^{50}?

17. The logarithm of a decimal fraction has a negative characteristic, a unit larger than the number of ciphers that follow the decimal point.

18. If $(1/2)^{500}$ be reduced to a decimal, how many ciphers follow the decimal point?

TABLES OF LOGARITHMS.

The logarithms of any set of consecutive numbers, arranged in a form convenient for use, constitute a *table of logarithms.* A table to the base 10 need give only the mantissas; the characteristics are evident. The difference of two consecutive mantissas is their *tabular difference.*

Tables may be carried to any number of decimal places; but the four-place table on pages 244, 245 is sufficiently accurate for ordinary use. The fourth figure is in error by less than half a unit. The first two figures of each number are printed at the left of the page, and the third figure at the top of the page, over the mantissa of the corresponding logarithm.

PROB. 4. TO TAKE OUT THE LOGARITHM OF A NUMBER.

(*a*) *A three-figure number: take out the tabular mantissa that lies in line with the first two figures and under the third;*

the characteristic is the exponent of that integer power of 10 *which lies next below the given number.*

If a number have one or two figures, make it three-figured by annexing zeros.

E.g., $\log 567 = 2.7536$; $\log 5.6 = 0.7482$; $\log .05 = \bar{2}.6990$.

(*b*) *A number of more than three figures: take out the tabular mantissa of the first three figures;*

subtract this mantissa from the next greater tabular mantissa;

multiply the difference so found by the remaining figures of the number as a decimal;

add this product, as a correction, to the mantissas of the first three figures.

E.g., to find log 500.6:

then $\because \log 500 = 2.6990$, $\log 501 = 2.6998$, [tables.

and $\log 501 - \log 500 = .0008$, $500.6 - 500 = .6$,

$\therefore \log 500.6 = 2.6990 + 6$ tenths of $.0008 = 2.6995$.

The rule for interpolation rests upon this property of logarithms, that their differences are nearly proportional to the differences of the numbers when the differences are small.

QUESTIONS.

1. How does a table of logarithms of prime numbers make it possible to find the logarithms of all other numbers?

2. Why do not tables of common logarithms contain characteristics? must they be given in tables to other bases?

3. Find $\log 2 - \log 1000$, giving a wholly negative logarithm as the result, and show that it is the same as $\log 1/1000 + \log 2$, or $\bar{3}.30103$.

4. In getting logarithms from a table, by what right and for what purpose are zeros annexed to numbers having fewer than three significant figures?

From the table take out the logarithms of:

5. 12.	6. 120.	7. 123.	8. 124.
9. 123.4.	10. 1.234.	11. 12350.	12. .001235.

13. In finding log 73265, to how large a difference in the number does the tabular difference correspond?

What part of this difference is the rest of the number?

Take out the logarithms of:

14. 9032.	15. .00064.	16. 75.15.	17. 6.872.
18. .25.	19. 2496000.	20. .00854.	21. 1000000.
22. 246.3.	23. .9467.	24. .007009.	25. 1463.

26. Show that the last paragraph in case (a) of prob. 4 might read: *make the characteristic one less than the number of figures in the integer part of the number.*

By use of the table on pp. 244, 245, find the logarithm of:

27. 43962.	28. 521.6701.	29. .004281.	30. 124365000.
31. 2.76314.	32. .4580012.	33. 6309.25.	34. .000519328.

35. Show how the logarithm of a number lying between two tabular numbers may be found from the larger of the two tabular logarithms.

36. Find in the table log 60, log 60.5, log 61, and show that, while the second of these logarithms is almost half-way between the other two, in numbers differing more widely, *e.g.*, 60, 70, 80, this proportion does not hold true.

PROB. 5. TO TAKE OUT A NUMBER FROM ITS LOGARITHM.

(*a*) *The mantissa found in the table: join the figure at the top that lies above the given mantissa to the two figures upon the same line at the extreme left;*

in the three-figure number thus found so place the decimal point that the number shall lie next above that power of 10 *whose exponent is the given characteristic.*

E.g., $\log^{-1} 2.7536 = 567$; $\log^{-1} 0.7482 = 5.6$; $\log^{-1} \bar{2}.6990 = .05$.

(*b*) *The mantissa not found in the table: take out the three-figure anti-logarithm of the tabular mantissa next less than the given mantissa;*

and to it join the quotient of the difference of these two mantissas by the tabular difference.

E.g., to take out $\log^{-1} 2.6995$:

then $\because$ the next less tabular mantissa is .6990, and the next greater .6998,

$\therefore$ the tabular difference is .0008;

i.e., an increase of .0008 in the mantissa .6990 corresponds to an increase of 1 in the number.

But $\because$ the given mantissa differs from .6990 by .0005,

$\therefore$ this difference corresponds to an increase in the anti-logarithm of .0005/.0008, *i.e.*, of .6.

and $\because \log^{-1} 2.6990 = 500$,

$\therefore$ the figures in the number sought are 5006, and the characteristic 2 shows that the number is 500.6.

So, to find $\log^{-1} \bar{3}.4986$:

then $\because$ the given mantissa lies between .4983 and .4997,

and $.4986 - .4983 = .0003$, $.4997 - .4983 = .0014$,
$.0003/.0014 = .2$,

$\therefore \log^{-1} .4986$ has the figures 3152,

and $\log^{-1} \bar{3}.4986 = .003152$.

The process is but the inverse of that for taking out logarithms, and the reason of the rule is the same for both.

LIMITATIONS IN THE USE OF THE TABLES.

The possible error of any logarithm, as printed in this table, is half a unit in the fourth place, and the possible error of any tabular difference is a unit; but the probable error is much less. The fourth figure of the anti-logarithm, the first got by division, is generally trustworthy; the fifth is rarely to be used. The possible error in the result is nearly ten times greater if the logarithm be near the end of the table than if near the beginning, for then the tabular difference, the divisor, is much smaller, and an error either in it or in the dividend has greater effect. If greater accuracy be desired, larger tables must be used.

QUESTIONS.

From the table take out the anti-logarithms of:

1. 1.0792.	2. 2.0792.	3. 2.0899.	4. 2.0934.
5. 2.0913.	6. 0.0913.	7. $\bar{4}$.0917.	8. $\bar{1}$.9652.

9. Show that the last paragraph of case (a) in prob. 5 might read: *make the number of figures in the integer part of the number one more than the characteristic of the logarithm.*

10. After dividing the difference between a given mantissa and the next less tabular mantissa by the tabular difference, why is the quotient annexed instead of added to the figures given by the table?

Find the anti-logarithms of:

11. 2.9053.	12. $\bar{1}$.7126.	13. .3402.	14. 1.4612.
15. 3.0024.	16. $\bar{2}$.5832.	17. $\bar{3}$.7368.	18. .5505.
19. 2.5337.	20. 2.6193.	21. $\bar{1}$.8000.	22. .0971.

23. If the logarithms of a series of multipliers and divisors be added, what logarithms must be regarded as negative?

If the sum of any column be negative it may be made positive by adding one or more tens to it, and subtracting the same number of units from the sum of the next column at the left.

24. Write under each other the logarithms of 3426, 4.003, .00162, 324.5, 1.64, and then so add them as to find the logarithm of $3426 \times 4.003 \times .00162 : 324.5 : 1.64$.

PROB. 6. TO DIVIDE A LOGARITHM WHOSE CHARACTERISTIC IS NEGATIVE.

Write down as the characteristic of the quotient the number of times the divisor is contained in that negative multiple of itself which is equal to, or next larger than, the negative characteristic;

carry the positive remainder to the mantissa and divide.

E.g., $\bar{4}.1234:3=(^{-}6+2.1234):3=\bar{2}.7078.$

So, $\bar{3}.4770\cdot 3/2=\bar{8}.4310/2 \qquad =\bar{4}.2155.$

PROB. 7. TO COMPUTE BY LOGARITHMS THE PRODUCTS, QUOTIENTS, POWERS, AND ROOTS OF NUMBERS.

For a product: add the logarithms of the factors, and take out the anti-logarithm of the sum.

For a quotient: from the logarithm of the dividend subtract that of the divisor, and take out the anti-logarithm.

For a power: multiply the logarithm of the base by the exponent of the power sought, and take out the anti-logarithm.

For a root: divide the logarithm of the base by the root-index, and take out the anti-logarithm.

E.g., to find the value of $(.01519\times 6.318:7.254)^{3/2}$:

NUMBERS.	LOGARITHMS.
.01519	$\bar{2}.1815$
$\times 6.318$	$+0.8006$
$:7.254$	-0.8605
	$\bar{2}.1216\times 3/2$
	$\bar{3}.1824$

and the number sought is 0.001522.

PROB. 8. TO SOLVE THE EXPONENTIAL EQUATION $A^x=B$.

Divide the logarithm of B *by the logarithm of the base* A.

The quotient is x, the exponent sought.

For $\because A^x=B$,

$\therefore x\cdot\log A=\log B$, whatever system of logarithms be used,

and $\quad x=\log B/\log A.$ Q. E. D.

QUESTIONS.

By logarithms, find the value of:

1. 575.25×1.06^{20}.
2. 575.25×1.03^{40}.
3. $\sqrt[5]{.00010098}$.
4. $[76^{3/2} \cdot 45^{1/4}]^{2/5}$.
5. $\dfrac{2^2 \cdot 5^3 \cdot 85^2}{3^2 \cdot 7^3}$.
6. $\dfrac{\sqrt{(97^2 - 9^2)}}{81 \cdot \sqrt[3]{572}}$.
7. $\dfrac{\sqrt{12} \cdot \sqrt[3]{65}}{\sqrt{5} \cdot \sqrt[7]{.18}}$.
8. $\dfrac{\sqrt[3]{83.64}}{.08145^2}$.
9. $\sqrt[7]{.0000000037591}$.
10. $\sqrt{236 \cdot 140 \cdot \sqrt[3]{96} : 215}$.

11. What power is 2 of 1.05? 3 of 1.04? 4 of 1.03? 5 of 1.02?

12. Given the logarithms of A, B, C, D, show how to find the value of $(-\text{A})^2 \cdot (-\text{B})^3 / -4\text{C} / -\text{D}$.

13. Divide the logarithm $\bar{3}.2614$ by 2, by 5, and by 6, in turn.

14. Find the cube root of $\log^{-1} \bar{7}.3550$.

15. Find the cube of the fourth root of $\log^{-1} \bar{6}.5448$.

Find by logarithms:

16. The simple interest of \$23.65 for 25 yr. 3 mo. at $7\frac{1}{4}$ %.

17. The amount of \$246 for $12\frac{1}{2}$ years if the interest be compounded annually at 8 per cent; if half-yearly at 4 per cent; if quarterly at 2 per cent.

18. The 20th term of a geometric series if $a = 5$, $r = 1\frac{1}{2}$.

19. If the number of births each year be one in forty-five of the population, and of deaths one in sixty, in how many years will the population double, taking no account of other sources of increase or decrease? triple? quadruple?

20. From the formula for compound interest, $a = p \cdot (1 + r)^t$, find an expression for t in terms of a, p, r.

21. How long must \$1000 be at compound interest to amount to \$1191.02 at 6 per cent a year? at 3 per cent half-yearly? at $1\frac{1}{2}$ per cent quarterly?

22. In a geometric series, given $l = ar^{n-1}$, then
$n - 1 = \log_r l/a$, $n = 1 + \log_r l - \log_r a = 1 + (\log l - \log a) : \log r$.

23. In a geometric series, $a = 5$, $l = 1280/6561$, $r = 2/3$: find n.

24. Solve the equations $(5\frac{4}{9})^x = 12\frac{19}{27}$; $a^{x+1}/b^{x-1} = c^{2x}$,

N	0	1	2	3	4	5	6	7	8	9	Dif.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	19
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	9
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	Dif.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

§ 6. QUESTIONS FOR REVIEW.

Define and illustrate:

1. A series; an arithmetic series; a geometric series; a harmonic series.

2. A rising series; a falling series; a continuous series.

3. The five elements of an arithmetic series; of a geometric series; the four elements of a harmonic series.

4. Arithmetic means; geometric means; harmonic means.

5. The major terms of a series; the minor terms.

6. A commensurable power; an incommensurable power.

7. A base; a logarithm; an anti-logarithm.

8. A system of logarithms; a table of logarithms.

9. The characteristic of a logarithm; the mantissa.

10. Natural logarithms; common logarithms.

11. Tabular numbers; tabular logarithms; tabular differences.

Write and prove the formulæ for:

12. The last term of an arithmetic series; the sum.

13. The last term of a geometric series; the sum.

14. The last term of a harmonic series.

15 The sum of an infinite falling geometric series; the value of a repeating decimal.

16. The arithmetic mean of two numbers; the geometric mean; the harmonic mean.

State the analogies of the three progressions.

17. The difference of any two terms of an arithmetic series.

18. The ratio of any two terms of a geometric series.

Give a general rule, with reasons and illustrations, for:

19. Finding the other two elements of an arithmetic series, when any three of them are given; of a geometric series.

20. Inserting means in an arithmetic series; in a geometric series; in a harmonic series.

State the principle, with proof, that relates to:

21. The product of incommensurable powers of the same base; the quotient of two such powers.

22. An incommensurable power of an incommensurable power of a base.

23. The product of like incommensurable powers of different bases; the quotient of two such powers.

---

24. What is the logarithm of unity to any base? of the base itself? of zero?

25. If the base be positive and larger than unity what is the logarithm of a number smaller than unity? of one larger than unity? How does the logarithm change as the number increases?

26. If the base be positive and smaller than unity what is the logarithm of a number smaller than unity? of one larger than unity? How does the logarithm change as the number increases?

27. What is the logarithm of the product of two numbers? of the quotient of two numbers? of the reciprocal of a number?

28. What relation has the logarithm of a power of a number to that of the number? the logarithm of a root?

29. What relation have the logarithms of two numbers, each taken to the other as base?

30. What relation have the natural and the common logarithms of the same number?

Give the general rule, with reasons and illustrations, for:

31. Taking out a logarithm from the table, when the number is found in the table; when not so found.

32. Taking out a number, from its logarithm, when the logarithm is found in the table; when not so found.

---

33. Define the characteristic and the mantissa of a logarithm, and show what relation the characteristic has to the position of the decimal point in the number.

IX. PERMUTATIONS, COMBINATIONS, AND PROBABILITIES.

The different groups that can be made of two or more things, without regard to order, are their *combinations;* the different groups that can be made of them, their order being considered, are their *permutations.*

Two permutations are different when either the things themselves are different or their order of arrangement is different; but two combinations are different only when at least one of the things contained in one of them is not found in the other.

E.g., *ab, ba, ac, ca, bc, cb,* are the six permutations of the three letters a, b, c, taken two at a time;
but *ab, ba* are the same combination, *ac, ca* are the same, and *bc, cb* are the same,
and in all, there are but three combinations.

So, *abc, bac, acb, cab, bca, cba* are the six permutations of the three letters a, b, c, taken all together;
but there is only one combination.

So, of four letters, a, b, c, d, taken three at a time, there are four combinations, *abc, abd, acd, bcd*;
and of each of them can be made six permutations, as above.

Nearly all investigations as to permutations and combinations depend upon the following self-evident principle:

Ax. *If there be a group of m things and a group of n things so related that either of the m things may be taken at random, and then either of the n things may be joined to it, there are mn ways in which such pairs may be made up, and no more.*

E.g., if a boy have 5 apples and 6 peaches, he may form 30 different pairs of them, an apple and a peach;
for the first apple may go with either of the 6 peaches, and so with the others.

So, if 5 men enter a room with 6 chairs the first man has choice of 6 chairs, the second man of the other 5 chairs, the third man of the other 4 chairs, and so on;

and, while the first man can be seated in but six ways, the first two men can be seated in $6 \cdot 5$ ways, the first three men in $6 \cdot 5 \cdot 4$ ways, and so on.

QUESTIONS.

1. Which of the groups below are permutations and which are combinations?

the three-figure numbers that can be made from n figures,
the products of four factors, taken from ten given factors,
the parties of four that can be made up from six men,
the ways twelve men can stand in a row, or in a ring,
the ways of dividing six things among three men.

2. What are the three combinations, in pairs, of a, b, c?

Make two permutations of each of these combinations.

So, of each of the combinations *abc, abd, acd, bcd* make six permutations, thus forming the twenty-four permutations of $a, b, c, d,$ taken in groups of three.

3. If permutations are to be made of the letters $a, b, c, d,$ how many choices are there for the first place?

Make all possible permutations of two by annexing to each of these first letters the other three letters in turn: how does the number of choices compare with that for the first place?

To each of the couplets annex the two remaining letters in turn, and form all the possible groups of three: how many letters remain to annex to each group of three?

4. How does the axiom apply in finding the number of couplets? in finding the number of threes?

5. Of the letters of the word *thing,* make all the possible permutations of two letters; of three letters; of four letters.

6. Why can more permutations be made with the same letters arranged in a row than in a ring?

7. Find the sum of all the four-figure numbers that can be expressed by the figures 1, 2, 3, 4.

If all these numbers be written one under another, how many times is each figure found in each column?

Show that the sum of the numbers formed as above is divisible by the sum of the four figures involved.

§ 1. PERMUTATIONS.

THEOR. 1. *The number of permutations of n things, all different, taken r at a time, is* $n(n-1)(n-2)\cdots(n-r+1)$.

For of n things taken singly, there are n choices and no more;
i.e., $\text{P}_1 n = n$;
so, if each of the n things be followed in turn by each of the $n-1$ things that remain, there are formed $n(n-1)$ couplets, all different,
i.e., $\text{P}_2 n = n(n-1)$;
so, if each of these $n(n-1)$ couplets be followed in turn by each of the $n-2$ things that remain, there are formed $n(n-1)(n-2)$ threes, all different,
i.e., $\text{P}_3 n = n(n-1)(n-2)$;
and so for the groups of four, of five, $\cdots$ of r,
i.e., $\text{P}_r n = n(n-1)(n-2)\cdots(n-r+1)$. Q.E.D.

COR. 1. *The number of permutations of n things all different, taken all together, is the product of all the integers from 1 to n, inclusive.*

For $\because$ here r is n, and $n-r+1$ is 1,
$\therefore$ $\text{P}_n n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

This product is indicated by the symbol $\underline{|n}$ or $n!$, and it is read *factorial n*.

COR. 2. *If in each group of r things some, or all, may be alike* (permutations with repetition), *then the number of such permutations is n^r.*

For $\because$ there is a choice of n things for the first place, then of n things for the second place, and so on,
$\therefore$ $\text{P}_{r,\,\text{rep.}}\, n = n\cdot n\cdot n \cdots r$ times $= n^r$.

THEOR. 2. *The number of permutations of n things, taken all together, when p things are alike, q things alike, r things alike, and so on, is* $n!/p!\,q!\,r!\cdots$

For $\because$ $p!$ permutations of the n things are formed by interchanging any p things among themselves, while the other things stand fast, if the p things be all different, but only one permutation if they be alike,

∴ the whole number of permutations is $p!$ times larger if the p things be all different than if they be alike;

and ∵ $P_n n = n!$ if the n things be all different, [th. 1 cr. 1.

∴ the number of permutations of n things, p alike, is $n!/p!$.

So, if q things be alike, r things alike, and so on,

∴ $P_{n.\, p \text{ alike}, q \text{ alike}, r \text{ alike} \dots} n = n!/p!\, q!\, r! \cdots$ Q. E. D.

QUESTIONS.

1. In making up permutations, why are there fewer choices for each successive place to be filled? at what rate does the number of such choices decrease?

What relation has the number of places filled, at any stage of the process, to the number of choices for the next place?

2. Why is the number of permutations of n things taken $n-1$ at a time the same as the number taken n at a time?

3. Find the number of permutations of 10 things, all different, 3 at a time; 5 at a time; 7 at a time; all together.

4. How many permutations, 3 letters at a time, can be made up from the word *mucilage*? from the word *formula*?

5. Of how many unlike things, taken all together, are there 720 permutations? 5040? 40320?

6. What is the value of $n!/(n-1)!$? of $n!/(n-2)!$? of $n!/(n-3)!$? of $n!/(n-r)!$? of $n!/r!$

7. The number of permutations of n things, r at a time, plus r times their number $r-1$ at a time, is

$n(n-1)\cdots(n-r+2)(n+1)$, *i.e.*, $P_r n + r \cdot P_{r-1} n = P_r(n+1)$.

8. In how many ways can 8 men stand in a row? n men?

In how many ways can 8 men sit around a table? n men?

9. In how many ways can 5 prizes be given to 5 boys?

10. A man has three ways of going to his place of business; in how many different ways can he plan his route for six days?

11. Find the number of permutations of ten flags if three be red and seven blue? if two be red, three white, five blue?

12. How many permutations can be formed from the word *London*? *Washington*? *Mississippi*? *Constantinople*?

§ 2. COMBINATIONS.

THEOR. 3. *The number of combinations of n things all different taken r at a time, is the quotient of the number of such permutations divided by r!*

For ∵ any combination of r different things gives $r!$ permutations of r things, [th. 1 cr. 1.

$\therefore \mathrm{C}_r n = \mathrm{P}_r n / r!$ Q. E. D. [th. 1.

COR. 1. *The number of combinations of n things taken n−r at a time, is the same as their number taken r at a time.*

For each group of r things leaves a group of $n-r$ things.

COR. 2. *The number of combinations of n things is greatest when r is the integer nearest in value to* $\frac{1}{2}n$.

For ∵ $\mathrm{C}_1 n = \frac{n}{1}$, $\mathrm{C}_2 n = \frac{n}{1} \cdot \frac{n-1}{2}$, $\mathrm{C}_3 n = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, ...

$$\mathrm{C}_r n = \frac{n}{1} \cdot \frac{n-1}{2} \cdots \frac{n-r+1}{r}, \quad \mathrm{C}_{r+1} n = \frac{n}{1} \cdots \frac{n-r+1}{r} \cdot \frac{n-r}{r+1},$$

$\therefore \mathrm{C}_r n$ is greatest when $(n-r+1)/r > 1 > (n-r)/(r+1)$,

i.e., when $\frac{1}{2}(n+1) > r > \frac{1}{2}(n-1)$, if n be even; Q. E. D.

and $\mathrm{C}_r n$, $\mathrm{C}_{r+1} n$ are equal to each other, and greater than the other terms of the series when

$(n-r+1)/r > 1 = (n-r)/(r+1)$,

i.e., when $r = \frac{1}{2}(n-1)$, if n be odd. Q. E. D.

COR. 3. *If in each group of r things some, or all, may be alike* (combinations with repetitions) *then the number of such combinations of n things is* $n(n+1)(n+2) \cdots (n+r-1)/r!$

For, let $a, b, c, \cdots l, m$, be any $n+1$ different things, form all possible groups of two, $(n+1) \cdot n/2!$ in all,

and replace am by aa, bm by bb, cm by cc, $\cdots$ lm by ll;

then m vanishes, and there result $n(n+1)/2!$ combinations, with repetitions, of the n things $a, b, c, \cdots l$.

So, let $a, b, c, \cdots l, m, n$ be any $n+2$ different things, form all possible groups of three, $n(n+1)(n+2)/3!$ in all,

and replace *amn* by *aaa*, *bmn* by *bbb*, $\cdots$, *lmn* by *lll*,
abm by *aba*, *acm* by *aca*, $\cdots$, *lkm* by *lkl*,
abn by *abb*, *acn* by *acc*, $\cdots$, *lkn* by *lkk*;

then m, n vanish, and there result $n(n+1)(n+2)/3!$ combinations, with repetitions, of the n things, $a, b, c, \cdots l$,

and so for groups of four things, of five things, $\cdots$ of r things.

QUESTIONS.

1. How many things are taken at a time, if the number of permutations and of combinations be the same?

2. Take the letters a, b, c, d, e and join to each of them every letter that follows it in the list, thus making all the groups of two; form the threes by joining to each couplet every letter that follows all its elements; so, the fours; the fives.

3. How many triangles can be formed by joining three vertices of a polygon of n sides? how many pentagons by joining five vertices? With six points construct the fifteen pentagons.

4. If an indefinite line be cut at four points, how many segments are formed? at six points? at n points?

5. Show that the formula for the number of combinations of n things taken r at a time, may be written $n!/r!(n-r)!$ Hence prove that $\mathrm{c}_r n = \mathrm{c}_{n-r} n$.

6. Find the number of combinations of 10 things all different taken 3 at a time; 5 at a time; 7 at a time.

The number of groups 3 at a time is the same as their number 7 at a time; and their number 5 at a time is greatest of all.

7. If there be four straight lines in a plane, no two parallel and no three meeting in a point, how many triangles are formed? if five lines? if n lines?

8. If there be six points in a plane, no three colinear, and lines join them so as to form the greatest possible number of figures of the same kind, what will those figures be?

9. If there be seven points in a plane, and they be joined so as to make triangles and quadrangles, of which sort is there the greater number? Draw the quadrangles.

10. Apply cor. 3 to find the number of terms in $(a+b)^n$.

THEOR. 4. *If there be n things all different, and if p, q, be any positive integers whose sum is n, then there are $n!/p!q!$ ways in which these n things can be made up into sets of p things and sets of q things.*

For a group of q things is left for every group of p things taken,
and $\quad {}_{C_p}n = n!/p!(n-p)! = n!/p!q!$ Q.E.D. [th. 3 cr. 1.

COR. 1. *If $n=p+q+r$, the number of sets of p things, q things, and r things, is $n!/p!q!r!$* [above.

For $\because$ n things give $n!/p!(q+r)!$ sets of p things and the same number of sets of $q+r$ things,

and $\because$ any set of $q+r$ things gives $(q+r)!/q!r!$ sets of q things and the same number of sets of r things,

$\therefore$ the whole number of sets is
$n!/p!(q+r)! \times (q+r)!/q!r!$, *i.e.*, $n!/p!q!r!$ Q.E.D.

So, when n is the sum of more than three integers.

E.g., if 12 recruits be divided into squads of three, four, and five, the number of such squads is $12!/3!4!5!$;
if divided into three equal squads and sent to different companies, their number is $12!/(4!)^3$.
if simply divided into three equal squads, $12!/3!(4!)^3$.

COR. 2. *The number of combinations of n different things in sets of p things, q things, and so on, when $n=p+q+\cdots$, equals the number of permutations of n things taken all together, when p things are alike, q things alike, and so on.*

For the $p!$ permutations of p different things are but a single combination,

and, if the p things be like things, but a single permutation.

COR. 3. *The value of the quotient $n!/p!q!r!\cdots$ is greatest when no one of the numbers p, q, r, $\cdots$ exceeds any other of them by more than a unit.*

For, let $p=q+2$; [hyp.

then $\because$ $p!q! = (q+2)!q! = (q+1)!(q+2)q! > (q+1)!(q+1)!$,

$\therefore$ the divisor is smallest and the quotient largest when p, q, r $\cdots$ are nearest to equality.

QUESTIONS.

1. There are as many combinations of n things taken r at a time as there are permutations of n things, all taken, when r things are of one kind and $n-r$ things of another.

2. A cent, a dime, a quarter of a dollar, a half dollar, and a dollar are each claimed by two boys: in how many different ways can the coins be divided between them?

3. In the proof of theor. 4, why is the number of combinations in sets of p things and q things the same as the number when p things are taken at a time?

4. Prove theor. 4 cor. 1 when $n=p+q+r+s+t$.

5. Whatever n may be, three numbers p, q, r can be found whose sum is n and which differ from each other by not more than a unit; but if n be a multiple of 3 and p, q, r be not taken all equal, then the difference between some two of them is at least 2. Generalize this proposition.

6. Show that the number of permutations that can be made from all of $2n$ things that are of two kinds is greatest when there are n things of each kind.

7. How can 18 things be divided among 5 persons in the greatest number of ways, each person receiving the same number of things at each distribution?

8. In the expansion of $(x+y)^{14}$, what term has the greatest coefficient? what term of $(x+y)^{21}$?

9. Of the combinations of eight letters a, b, c, $\cdots$, taken four at a time, how many contain a? not a? both a and b? a and not b? neither a nor b? a, b, and c? a or b or c? neither a nor b nor c?

10. In finding the product $(x+a)\cdot(x+a)\cdots n$ factors, the various terms in the several partial products are all the possible permutations of n letters taken all together, wherein part are a's and the rest x's.

The coefficient of x^n is $\mathrm{P}_n n_{n \text{ alike}}$; that of $x^{n-1}a$, $\mathrm{P}_n n_{n-1 \text{ alike}}$; that of $x^{n-2}a^2$, $\mathrm{P}_n n_{n-2 \text{ alike}, 2 \text{ alike}}$; and so on.

Hence prove the binomial theorem.

Theor. 5. *If there be n sets of things, the first set containing p things, the next q things, and so on, and if combinations of n things be made up by taking one thing from each set, then the number of such combinations is the product* $p \cdot q \cdot r \cdots$

For each of the p things may be joined to each of the q things,
each of these pq pairs to each of the r things, and so on.
Q.E.D. [ax.

Cor. 1. *The number of combinations made by taking h of the p things, j of the q things, k of the r things, and so on, is the product of the number of combinations of p things taken h at a time, of q things taken j at a time, and so on.*

Cor. 2. *With the data of cor. 1, the number of permutations possible is* $\text{c}_h p \cdot \text{c}_j q \cdot \text{c}_k r \cdots (h+j+k+\cdots)!$

Cor. 3. *The number of possible combinations of some or all of* $p+q+r+\cdots$ *things, of which p things are alike, q things alike, and so on, is* $(p+1)\cdot(q+1)\cdot(r+1)\cdots-1$.

For the p things may be treated in $p+1$ ways;
i.e., none, or one, or two, $\cdots$ or p of them may be taken,
and each of these $(p+1)$ dispositions of the p things may be joined to each of the $(q+1)$ dispositions of the q things,
each of these pairs may be joined to each of the $(r+1)$ dispositions of the r things, and so on,
and the whole number is $(p+1)\cdot(q+1)\cdot(r+1)\cdots$;
but $\because$ this includes the case when no thing is taken from any group, and this case can not be counted,

$\therefore \text{c}=(p+1)\cdot(q+1)\cdot(r+1)\cdots-1$.

Cor. 4. *The number of possible combinations of n different things, taken some or all at a time, is* 2^n-1.

For $\because$ each thing may be either taken or left,
and either disposition of one thing may be followed by either disposition of every other thing,

$\therefore$ the whole number of combinations including that where no thing is taken is $2\cdot2\cdot2\cdots n$ times, $=2^n$. [ax.

$\therefore \text{c}=2^n-1$. Q.E.D.

QUESTIONS.

1. Write out all the measures, prime and composite, of 6; of 30; of 2310; of abc; of $abcd$; of a^4-x^4.

2. Out of 12 democrats and 16 republicans how many committees can be made up, each consisting of 3 democrats and 4 republicans? how many committees of seven can be made up with the condition that each committee shall contain at least two men of each party?

3. How many different signals can be made by hoisting 6 differently colored flags one above another, when any number of them may be raised at once?

4. If $a, b, c, \cdots$ be n different prime numbers: find the number of different measures of the product $a^n \cdot b^{n-1} \cdot c^{n-2} \ldots$

5. If all groups of letters were words, how many words composed of two consonants and one vowel could be made from our alphabet of five vowels and twenty-one consonants?

6. The number of permutations of n things of two kinds taken n at a time, when some or all are alike, is the same as the number of combinations of n different things, some or all at a time; hence the sum of the coefficients of $(a+x)^n$ is 2^n.

7. From six apples, five pears, and four plums, how many selections of an apple, a pear, and a plum can be made?

8. From a party of six ladies and seven gentlemen, how many groups each of four ladies and four gentlemen can be formed? how many sets of four couples for a quadrille?

9. Given m things of one kind and n things of another, find how many permutations can be made by taking r things of the first set and s things of the second.

10. How many different sums of money can be formed from a cent, a half-dime, a dime, three half-dollars, five dollars?

11. A signal staff has five arms, each of which may assume four positions: how many signals can be made?

12. If of $p+q+r$ things, p things be alike, q things alike, and the rest all different, the whole number of combinations possible is $(p+1)\cdot(q+1)\cdot 2^r-1$.

§ 3. PROBABILITIES.

The theory of probabilities is concerned with classes of things about whose individuals there is uncertainty; it might well be called the *doctrine of averages*. It seeks to show what likelihood there is that a particular event may happen or fail, the basis of computation being such known facts as these:

that the event considered must happen once, or a fixed number of times, in a given number of possible cases;

that it cannot happen more than such number;

that in the past such events have happened with such, or such, a degree of frequency.

E.g., if the twenty-six letters of the alphabet be written singly on cards of the same size and shape, and these cards be thrown into a box;

then, in twenty-six successive drawings, each letter will be drawn out once, and but once, and at the first drawing one letter is as likely to come out as another.

So, as shown by the Institute of Actuaries' tables, of 100,000 healthy boys of ten 96,223 have reached the age twenty, and 609 have died between twenty and twenty-one; 1460 have reached ninety, and 408 have died within a year thereafter; and what has happened in the past may be expected in the future with ratios very slightly changed.

The *probability* of an event is the ratio of the number of cases in which the event happens, *favorable cases*, to the whole number of cases considered.

E.g., in the example above, the probability that the letter A be drawn out, at the first drawing, is the ratio 1/26; that A be not drawn, 25/26; that one of the five vowels be so drawn, 5/26; that neither of the vowels be drawn, 21/26; that one of the consonants be drawn, 21/26.

So, barring special conditions of health and occupation, the probability that a certain boy of ten live to the age of twenty is .96223; to the age of ninety, .0146; that he die before twenty, .03777; before ninety, .9854.

QUESTIONS.

1. If the twenty-six letters of the alphabet be written on separate cards, what is the probability

that x be first drawn? that x be not drawn?

that either x or y be drawn? that neither x nor y be drawn?

that either x or y or z be drawn? that neither x nor y nor z be drawn?

2. How many different pairs of letters are there? [th. 3.

If two letters be drawn at a time what is the probability of drawing a particular pair? of not drawing that pair?

3. How many ways are there of drawing two letters in succession? three letters? four letters? [th. 1.

What is the probability of drawing first A, then B? A, B without regard to order? A, then B, then C? A, B, C without regard to order? A, then B, then C, then D? A, B, C, D without regard to order?

4. If of men of A's age one in eight die in five years thereafter, and of men five years older one in seven die, what is the probability that A will die within five years? that he will live at least five years? that he will die within ten years? that he will live ten years? that he will die within fifteen years?

5. If of 588 men of sixty, 17, on an average, die in a year, of the 571 men of sixty-one 18 die; of the 553 men of sixty-two 19 die; of the 534 men of sixty-three 20 die; and of the 514 men of sixty-four 21 die, what is the probability that a man of sixty lives one year? two years? three years? four years? five years? that a man of sixty-one dies within a year? two years? three years? four years? that a man of sixty-two lives till he is sixty-five? that he dies before he is sixty-five?

6. In throwing a single die what is the probability of a six? not a six? a six or an ace? neither a six nor an ace?

7. If a boy with three red marbles in his pocket, five white, and seven blue ones, take out one at random, what is the probability that it is blue? not blue? a particular blue one? not that blue one? red? white? either red or white?

SIMPLE PROBABILITIES.

If the probability of a single event be considered, it is the *simple probability* of the event; and the sum of the probabilities for and against such event is always unity, *i.e.*, certainty.

E.g., that the letter A be drawn out at the first drawing the probability is 1/26, that it be not drawn 25/26,

and $1/26 + 25/26 = 1$.

THEOR. 6. *If there be two or more events that are mutually exclusive, the probability that some one of them shall happen is the sum of their separate probabilities.*

E.g., the probability that some one of the five vowels shall be drawn is five-fold that of one of them.

The probability of an event is not the same to every man; to each one it depends on his knowledge of the facts.

E.g., To one who knows, of a horse, only that he has been a frequent winner, the probability of his winning at a race to-day may be high;

but to the groom, who knows that he has gone lame, his defeat is almost certain.

PROBABLE VALUES.

The *probable value* of a sum of money payable at some future time on conditions whose fulfilment is uncertain is the product of the sum due by the probability of receiving it.

This principle is of special importance in life-insurance.

E.g., to find the cost of insuring a man of twenty for a year:

Were death during the year certain and payment made at its end, the premium would be the present worth for a year, at an agreed rate of interest, of the face of the policy; and at four per cent the premium on \$1000 would be \$961.54;

but ∵ on an average, 609 men out of 96223 die between twenty and twenty-one,

∴ the probability of death, and of the consequent payment of the money, is 609/96223,

and $\$961.54 \times 609/96223 = \$6.086$, the net cost for \$1000.

QUESTIONS.

1. If each letter of the alphabet when drawn out be replaced before another drawing, how many possible ways are there of drawing two letters in succession? three letters? the same letter twice? three times? [th. 1 cr. 2.

2. With two dice what is the probability of throwing a four and a three? a double four? a three, a four, and a five, with three dice? three fours?

3. What is the probability of throwing exactly 10 in a single throw with three dice? 12? 15? 18? 20? less than 8?

Are the probabilities the same to a bystander as to a player who has honest dice? loaded dice?

4. If a man know that he is to receive a sum of money that is expressed in dollars by a three-figure number made up of the digits 3, 5, 7, but know not the order of the digits, what is the value of his expectation?

5. A friend is one of two hundred passengers on a ship that carries a crew of a hundred men, and it is reported that one man was lost during the voyage; what is the probability, to me, that it was my friend? later it is reported that it was a passenger; what is the probability now? and still later the name Johnson is given, my friend's name, but there were three Johnsons aboard; what now? what, to the ship's surgeon?

6. If of 1000 boys of ten 956 live to be twenty-one, what is the present value of $10,000 to be paid on his twenty first birthday to a boy now ten, the amount of $1 at compound interest for eleven years, at four per cent, being $1.53945?

So, if the money earn five per cent, the amount of $1 for eleven years being $1.71034?

So, if it earn but three per cent, the amount being $1.38423?

7. If p stand for one payment, r for the rate of interest, l_1, l_2, l_3, l_4, for the probabilities of living one, two, three, four years; find v, the present value of an annuity to run four years, or till previous death.

Make the problem general by writing n years, and $l_1, l_2, \cdots l_n$ for the probabilities of living one, two, $\cdots$, n years.

JOINT PROBABILITIES.

The probability of the simultaneous occurrence of two or more independent events is their *joint probability.*

E.g., that the letter A be drawn and an ace be thrown.

THEOR. 7. *If there be two or more independent events such that the simple probability of the first is* m/n, *that of the second,* m'/n', *and so on, then their joint probability is the product* $m/n \cdot m'/n' \cdots$

For ∵ the first event happens m times out of n, the second m' times out of n', and so on, [hyp.

∴ of any nn' joint events, mn' are favorable to the first event,

and of these mn' joint events favorable to the first event, mm' are favorable to the second event,

i.e., of nn' joint events, mm' are favorable to both events,

∴ the joint probability of the two events is mm'/nn';

and so if there be three or more events. Q.E.D.

E.g., that the letter A be drawn and an ace be thrown, the probability is $1/26 \cdot 1/6$,

that A be drawn and an ace not thrown, $1/26 \cdot 5/6$,

that A be not drawn and an ace be thrown, $25/26 \cdot 1/6$,

that A be not drawn and an ace not thrown, $25/26 \cdot 5/6$;

and the sum of all these products is unity.

So, the probability that A be first drawn and then B is $1/26^2$ if A be replaced after the first drawing,

and it is $1/26 \cdot 1/25$, *i.e.,* $1/650$, if A be not replaced.

So the probability that A, aged ninety, and B, aged twenty, shall both die within a year is $408/1460 \cdot 609/96223$,

that both live the year, $1052/1460 \cdot 95614/96223$,

that A lives and B dies, $1052/1460 \cdot 609/96223$,

that A dies and B lives, $408/1460 \cdot 95614/96223$;

and the sum of all these products is unity.

QUESTIONS.

1. If a bag hold three red balls, five white, and seven blue ones, find the probability of drawing three red balls in succession. Show that this probability is the same as that of drawing the three red balls all at once.

2. A bag holds m white balls and n black ones; the probability of drawing first a white ball and then a black one, and so on till all the balls left are of one color, is the same as that of getting all the white balls in a single drawing of m balls.

3. A man on a journey must make four connections to get through in time: if the probability of making each connection be 3/4, what is the probability of making them all?

4. A man of thirty marries a wife of twenty-five: if of 93 persons of twenty-five 90 reach thirty, 26 reach seventy-five, and 14 reach eighty, what is the probability of their celebrating a golden wedding?

5. On an average A speaks the truth three times out of four, and B nine times out of ten: what is the probability that both will assert a fact known to them both? that both will deny it? that their statements will be contradictory? that one or the other of these cases will occur?

6. The probability that A can solve a certain problem is 2/5, that B can solve it 2/3: if both try it, what is the probability of its being solved? what, that A succeeds and B fails? that A fails and B succeeds? that both succeed?

7. In one purse are ten coins, a sovereign and nine shillings; in another purse are ten coins all shillings; nine coins are taken from the first purse and put in the other, then nine coins are taken from the second purse and put in the first: what is the probability that the sovereign is still in the first purse?

8. If the probability that a ship will not meet a gale be 3/4; that, if it meet one, it will not be disabled, 5/6; that, if disabled, it will be kept afloat by the pumps, 1/2; that, if the pumps fail, the passengers will all escape by the boats, 1/3: find the probability of loss of life by shipwreck.

COR. 1. *If p be the simple probability of the occurrence of an event in one trial, then p^n is the probability that it occurs in all of n successive trials.*

E.g., the probability that the letter A be drawn twice in succession, being replaced after the first drawing, is $1/26^2$.

COR. 2. *If the probability of the occurrence of an event in one trial be p and of its failure q, then the probability of its occurrence r times in n trials is the $(n-r+1)$th term of the expansion of $(p+q)^n$.*

For the probability that the event occurs n times in succession is p^n, [cr. 1.

that it occurs $n-1$ times and fails once, is the product $p^{n-1}\cdot q$ taken as many times as there are permutations of n things with $n-1$ of them alike, *i.e.*, $n\cdot p^{n-1}q$, [th. 6.

that it occurs $n-2$ times and fails twice, is the product $p^{n-2}\cdot q^2$ taken as many times as there are permutations of n things with $n-2$ of them alike and 2 alike,

i.e., $\frac{1}{2}n(n-1)\cdot p^{n-2}\cdot q^2$, and so on;

that it occurs r times and fails $n-r$ times, is the product $p^r\cdot q^{n-r}$ taken as many times as there are permutations of n things with r of them alike and $n-r$ alike,

i.e., $n(n-1)\cdots(n-r+1)/r!\cdot p^r\cdot q^{n-r}$. Q.E.D.

COR. 3. *The probability that an event occurs at least r times in n trials is the sum of the first $n-r+1$ terms in the expansion of $(p+q)^n$.*

COR. 4. *That value of r for which the probability is greatest is the largest integer in $q(n+1)$.*

For the expression $n(n-1)(n-2)\cdots(n-r+1)/r!\cdot p^{n-r}\cdot q^r$

is greatest when $\dfrac{n-r+1}{r}\cdot\dfrac{q}{p}>1>\dfrac{n-r}{r+1}\cdot\dfrac{q}{p}$, [th. 3 cr. 2.

i.e., when $nq-rq+q>rp$, and $nq+q>r(p+q)$,

and when $rp+p>nq-rq$, and $r(p+q)>nq-p$;

and $\because p+q=1$, [hyp.

$\therefore$ it is greatest when $q(n+1)>r>q(n+1)-1$. Q.E.D

QUESTIONS.

1. If the probability that a man of fifty live to be eighty be 1/5, what is the probability that of six men of fifty, three at least reach eighty ? four at least ? five at least ? all of the six ?

2. If on an average, of the ships engaged in a certain trade, nine out of ten return safely, find the probability that out of five ships expected three come into port.

3. In how many trials is there an even chance of throwing double sixes with two dice ? a single six with one die ?

4. Two players A, B, of equal skill are interrupted in a game when A wants two points of winning and B three: show that the prize should be divided in the ratio 11/5.

5. The probability that a man will die within a year is 1/8; that his wife will die, 1/10; that his son will die, 1/60: if all three. or any two of them, be living at the end of the year they are to receive $10,000 in equal shares; what is the value of the expectation of each of them, interest at five per cent?

6. At simple interest, five per cent, find the present value of an annuity of $200 to run two years, and contingent on the joint lives of two persons whose probabilities of living for the next two years are 76/77, 74/75 for the first person, and 66/76, 65/75 for the other, the annuity being payable only if both be living; payable if either be living.

7. Three men A, B, C, throw a die alternately in the order of their names, and whoever first throws a five wins $182; show that their expectations are $72, $60, $50.

8. It is a question whether A has been elected; B tells C that D told him that A was elected, but C thinks it an even chance whether D said *elected* or *not elected*. A is elected if B and D both spoke truly or both falsely. find the probabilities.

9. If of thirteen aldermen at dinner the probabilities of living a year be 13/14, 14/15, 15/16, 16/17, 17/18, 18/19, 19/20, 20/21, 21/22, 22/23, 23/24, 24/25, 25/26; what is the probability that all of them live a year? what, that some one of them dies within a year ? what, if there be but twelve men?

§ 3. QUESTIONS FOR REVIEW.

Define and illustrate:

1. Permutations; permutations of n different things taken r things at a time; permutations with repetitions.

2. Combinations; combinations of n different things taken r things at a time; combinations with repetitions.

3. A factorial number.

4. The probability of an event; simple probability.

5. Probable values; the joint probability of two events.

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6. State the fundamental principle of permutations and combinations.

7. Show how to write out the permutations of n letters in groups of two letters; of three letters; $\cdots$ of r letters.

8. Show how to write out the combinations of n letters in groups of two letters; of three letters; $\cdots$ of r letters.

State and prove the general rule for finding:

9. The number of permutations of n things, all different,
in groups of r different things;
all together;
in groups of r things with repetitions allowed.

10. The number of permutations of n things, all together, with p things alike, q things alike, r things alike, and so on.

11. The number of combinations of n things, all different.
in groups of r different things;
in groups of r things with repetition allowed;
in groups of p things, q things, r things, and so on.

12. The number of combinations, in groups:
of one thing from each of n sets of things that contain p things, q things, r things, and so on;
of h things out of a set of p things, j things out of a set of q things, k things out of a set of r things, and so on;
of some or all of $p+q+r+\cdots$ things, when p things are alike, q things alike, r things alike, and so on;
of some or all of n different things.

13. The value of r that makes $C_r n$ the greatest.

14. The relations of $p, q, r, \cdots$ that give the greatest number of combinations of n things in sets of p things, q things, r things, and so on.

Prove that:

15. Of n different things there as many combinations in groups of $n-r$ things as in groups of r things.

16. Of n different things there are as many combinations in sets of p things, of q things, of r things, and so on, as there are permutations of n things taken all together, when p things are alike, q things alike, r things alike, and so on.

As deductions from the principles established in this chapter:

17. Prove the binomial theorem.

18. Find the number of terms in the expansion of a power of a binomial.

19. Find the term of the expansion whose value is greatest.

State and prove the rule for finding the probability:

20. That some one of n mutually exclusive events will occur.

21. That two or more events of known probability will occur jointly.

22. That an event will occur in all of n successive trials.

23. That it will occur exactly r times in n trials.

24. That it will occur at least r times in n trials.

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25. State and prove the rule for finding the most probable number of successes in n trials of things of known probability.

26. Show how far the doctrine of probabilities can be applied in any individual case; and where it fails.

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27. Three works, one of two volumes, one of three, and one of four, stand side by side: what is the probability that the volumes of each work stand in their proper order?

28. If $C_r 18 = C_{r+2} 18$, find $C_5 r$.

29. Find the whole number of combinations of $p+q+r$ things of which p things are alike, q things alike, and the rest all different.

30. Find the odds against A's winning four games before B wins two at a game where A is twice as good a player as B.

31. If $a, b, c, \cdots n$ be different prime numbers, what is the number of measures of the product $a^{14}\cdot b^{13}\cdot c^{12}\cdots l^{3}\cdot m^{2}\cdot n^{1}$.

32. Find the chance of throwing at least eight in a single throw with two dice; with three dice.

33. A and B play a set of games in which A's chance in each game is p, and B's q: show that the probability of A's winning m games out of $m+n$ games is

$$p^m\cdot[+np+n(n+1)p^2/2\,!+\cdots$$
$$+n(n+1)\cdots(n+m-2)p^{m-1}/(m-1)\,!].$$

34. From a bag that holds n balls a man draws out a ball and replaces it n times: what is the probability of his having drawn every ball in the bag?

35. Into a box having three equal compartments four balls are thrown at random: show that there are eighty-one possible arrangements; and find the probability that the four balls are all in one compartment; that three of them are in one compartment and one in another; that two of them are in one compartment and two in another; that two of them are in one compartment, and one in each of the others.

36. The number of combinations of n different things in groups of r things, with repetition, is the number of combinations of $n+r-1$ things in groups of r things without repetition.

37. The number of possible combinations of n things in groups the number of whose elements is even differs by one from the number of such combinations in groups the number of whose elements is odd.

INDEX.